Aalto University School of Electrical Engineering
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EXAM
11.3.2013

S-72.2410 Information Theory

1. (6p.) Entropy. Let the random variables X_1 and X_2 be identically distributed—that is, the probabilities of the outcomes coincide—but not necessarily independent. Now define

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}.$$

   (a) (4p.) Calculate ρ for the following joint distribution:

$X_2 \backslash X_1$	1	2	3
1	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$
2	0	$\frac{1}{16}$	$\frac{3}{16}$
3	$\frac{1}{4}$	0	0

   (b) (2p.+possible bonus point) What is the largest and the smallest value that ρ can get in general? Motivate your answer. You get 1 bonus point if you can formally prove the exact conditions under which the largest and smallest value are attained.

2. (6p.) Concepts and terminology. Define the following terms in a concise and precise way.

   (a) (2p.) lossy vs. lossless compression
   (b) (1p.) Gaussian channel
   (c) (1p.) multiple access channel
   (d) (1p.) mutual information
   (e) (1p.) universal coding

3. (6p.) Channel capacity. A ship can transmit information to another ship by putting up a flag.

   (a) (2p.) Assume that a ship has a set of four different flags and one at a time can be used. Determine the capacity in bits/second of this communication channel if it takes 5 seconds to get a flag up, it stays up for 10 seconds, and it takes yet another 5 seconds to get it down. (You must always take the flag down even if the same flag is used several times in a row.)

   (b) (2p.) Answer the same question as (a) under the assumption that all four flags are used at a time (and there is indeed just one flag available for each of the colors). The situation is exactly like in Fig. 1. The times are like in (a): it takes 5 seconds to get all flags up, etc.

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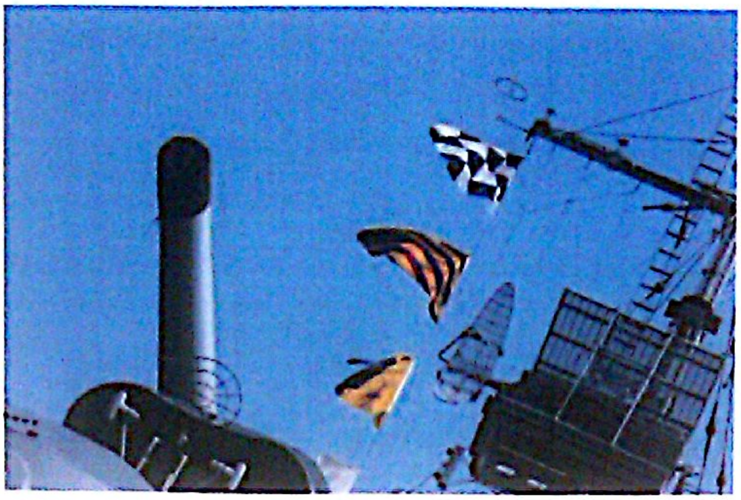

Figure 1: Signalling with flags.

(c) (2p.) Assume that two of the flags have red shades and two have blue shades, and during fog it is difficult to distinguish between the two blue shades and similarly for the red ones. What signalling system could you agree on to deal with this challenge? At what rate would you then be able to transmit information, in cases (a) and (b)? Obviously, you should try to maximize the rate.

4. (6p.) Source coding. Balls with different colors are drawn at random from a hat. The following table shows the possible colors and the probability mass function p of the random variable X:

	yellow	black	red	green	blue
p	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{12}$

(a) (1p.) Find the entropy $H(X)$.

(b) (3p.) Construct a uniquely decodable binary code of minimum expected length that encodes the colors of the balls. What is the minimum expected length?

(c) (2p.) Sally claims she has found a uniquely decodable binary code with the following codeword lengths:

yellow	black	red	green	blue
2	2	2	2	3

Show that Sally must be wrong. [Sally actually claimed that she has a strategy for guessing the color of a ball that will need the stated number of yes-no questions depending on the answer, but this is equivalent to the formulation above.]