Mat-1.1632 Mathematics 3-II

Examination 17.12.2012

Solutions

Problem 1

State the definitions of

(a) skew-symmetric matrix (1p)

Answer:

A square matrix **A** is called skew-symmetric if $\mathbf{A}^T = -\mathbf{A}$.

(b) orthogonal matrix (1p)

Answer:

A square matrix **A** is called orthogonal if $\mathbf{A}^{-1}=\mathbf{A}^T$.

(c) rank of matrix (1p)

Answer:

The rank of a matrix **A** is the highest order of nonzero minor(s).

(d) "eigenvalue" and "eigenvector" (1p)

Answer

Let **A** be a square matrix. A nonzero vector **x** is called an *eigenvector* of the matrix **A**, if $\mathbf{Ax}=\lambda\mathbf{x}$; λ is an *eigenvalue* of the matrix A, corresponding to this eigenvector.

(e) vector norm (1p)

Answer

A vector norm for the vector **x** is a number (denoted by $\|\mathbf{x}\|$) satisfying the following properties:

a) $\|\mathbf{x}\|$ is a nonnegative real number

b) $\|\mathbf{x}\|=0$ if and only if $\mathbf{x}=0$

c) $\|k\mathbf{x}\|=|k|\ \|\mathbf{x}\|$

d) $\|\mathbf{x}+\mathbf{y}\|\le\|\mathbf{x}\|+\|\mathbf{y}\|$.

(f) condition number (1p)

Answer:

The condition number κ(**A**) of a non-singular square matrix **A** is $\kappa(\mathbf{A}) = \|\mathbf{A}\|\cdot\|\mathbf{A}^{-1}\|$.

Problem 2

(a) Show that

$$\det\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\det\begin{pmatrix} e & f \\ g & h \end{pmatrix} \qquad (2p)$$

(b) Transform the quadratic form $Q(x_1,x_2) = -x_1^2 - x_2^2 + 4x_1x_2$ to the principal axis form. (4p)

Solution

(a) This directly is follow from the Laplace theorem.

Let us select 2 first rows. The only non-zero minor contained in these rows is $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Its cofactor is $\begin{vmatrix} e & f \\ g & h \end{vmatrix} \cdot (-1)^{3+4+3+4}$. Thus the determinant of the block matrix of this problem is equal to the determinant of the blocks, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det\begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

(b) The matrix of the quadratic form is $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$. Its eigenvalues are:

$$|\mathbf{A} - \lambda \mathbf{E}| = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 4 = \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1) = 0.$$

Eigenvalues are $\lambda_1=1$, $\lambda_2=-3$.

Thus, the principal axis form is $Q = \tilde{x}_1^2 - 3\tilde{x}_2^2$. In order to find the matrix of transformation to the new coordinates, we have to find eigenvectors.

$\lambda_1=1$. $(\mathbf{A}-\lambda\mathbf{E})\mathbf{X}_1=0$, or $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$. The eigenvector is $\mathbf{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the normalized eigenvector is $\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2=-3$, $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$. The eigenvector is $\mathbf{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, the normalized eigenvector is $\mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Therefore, the transformation to the new coordinates is $\mathbf{x} = \mathbf{C}\tilde{\mathbf{x}}$, $\mathbf{C} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

Problem 3

Find a general solution of the system $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$, $\mathbf{A} = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ (6p)

Solution.

The eigenvalues and eigenvectors are computed in Problem 5. Characteristic equation: $\det(\mathbf{A}-\lambda\mathbf{E})=0$, i.e. $\begin{vmatrix} 3-\lambda & 5 & 3 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda)(1-\lambda) = 0$.

Consequently, eigenvalues of this matrix are $\lambda_1=3$, $\lambda_2=2$, $\lambda_3=1$.

The eigenvectors can be found by solving the equation $(\mathbf{A}-\lambda_i\mathbf{E})\mathbf{x}=0$.

$\underline{\lambda_1=3}$ $\begin{bmatrix} 0 & 5 & 3 \\ 0 & -1 & 6 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$

From the 3rd equation, $x_3=0$. From the 2nd equation $-x_2+6x_3=0$, it follows that $x_2=0$ too. Therefore, we can choose $x_1=1$. Thus, the eigenvector $\mathbf{V}_1=[1\ 0\ 0]^T$.

$\underline{\lambda_2=2}$ $\begin{bmatrix} 1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$

From the 3rd equation, $x_3=0$. From the 1st equation $x_1+5x_2=0$, it follows that $x_1=-5x_2$. Consequently, the eigenvector $\mathbf{V}_2=[5\ -1\ 0]^T$.

$\underline{\lambda_3=1}$ $\begin{bmatrix} 2 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$

From the 2nd equation $x_2+6x_3=0$ we get $x_2=-6x_3$. From the 1st equation $2x_1+5x_2+3x_3=0$, it follows that $x_1=-5/2x_2-3/2x_3=-5/2(-6x_3)-3/2x_3=27/2x_3$. Consequently, the eigenvector $\mathbf{V}_3=[27\ -12\ 2]^T$.
Therefore, the general solution of the system of differential equations can be written as

$$y=C_1\begin{pmatrix}1\\0\\0\end{pmatrix}e^{3t}+C_2\begin{pmatrix}5\\-1\\0\end{pmatrix}e^{2t}+C_3\begin{pmatrix}27\\-12\\2\end{pmatrix}e^{t}$$

Problem 4

Find critical points of the system and determine their type and stability

$$\begin{cases}\dot{x}=y\\ \dot{y}=x-x^2\end{cases}$$

Solution.

From the equations $\dot{x}=\dot{y}=0$, or $\begin{cases}y=0\\ x(1-x)=0\end{cases}$, we find the critical points of the system.

They are (0,0) and (1,0).

Consider the point <u>(0,0).</u> The linearized at (0,0) system is $\begin{cases}\dot{x}=y\\ \dot{y}=x\end{cases}$.

Its matrix $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ has eigenvalues $\lambda_{1,2}=\pm 1$ (Found from the characteristic equation $\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=\lambda^2-1=0$). Therefore, the critical point (0,0) is a saddle point (unstable).

Consider the point <u>(1,0).</u> We linearized the system at the point (1,0). Applying the linearization formula $\begin{cases}\dot{x}=\left.\dfrac{\partial f_1}{\partial x}\right|_M\cdot(x-x_0)+\left.\dfrac{\partial f_1}{\partial y}\right|_M\cdot(y-y_0)\\ \dot{y}=\left.\dfrac{\partial f_2}{\partial x}\right|_M\cdot(x-x_0)+\left.\dfrac{\partial f_2}{\partial y}\right|_M\cdot(y-y_0)\end{cases}$ with $f_1=y$ and $f_2=x-x^2$, we get

$$\begin{cases}\dot{x} = y \\ \dot{y} = -1\cdot(x-1)\end{cases}, \text{ or } \begin{cases}\dot{\tilde{x}} = \tilde{y} \\ \dot{\tilde{y}} = -\tilde{x}\end{cases}, \quad \text{where} \quad \tilde{y} = y,\ \tilde{x} = x-1.$$

The matrix of the linearized system $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has eigenvalues $\lambda_{1,2} = \pm i$ (Found from the characteristic equation $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$). The eigenvalues are pure imaginary.

Therefore, the critical point (1,0) is a center (unstable).

Answer: (0,0) is a saddle point,(1,0) is a center

Problem 5

Consider the mixed problem for heat equation

(1) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

(2) $u(x,0)=f(x)$

(3) $u'_x(0,t) = 0$, $u'_x(1,t) = 0$

(a) Substituting $u(x,t)=X(x)T(t)$, derive the ordinary differential equations $X'' + \lambda^2 X = 0$ and $T' + \lambda^2 T = 0$, where $\lambda \in \mathbf{R}$ is a constant. (1p)

Solution.

Substituting $u(x,t)=X(x)T(t)$ into equation (1), we get $XT' = X''T$. To separate variables, we divide both sides by XT and equate to the constant $-\lambda^2$:

$$\frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda^2$$

This yields two ordinary differential equations:

$$X'' + \lambda^2 X = 0$$
$$T' + \lambda^2 T = 0.$$

(b) Find the function $X(x)$ satisfying the boundary conditions derived from Eq.(3), and the constant λ. Show that the solution to the problem (1),(3) can be presented in the form

(4) $$u(x,t) = B_0 + \sum_{k=1}^{\infty} B_k \exp(-\pi^2 k^2 t)\cos(\pi k x)$$ (2p)

Solution.

We want to find the solution of

(*) $$X'' + \lambda^2 X = 0,$$

satisfying the boundary conditions $X'(0) = X'(1) = 0$. These boundary conditions are obtained from (3) under assumption $T(t)\neq 0$.

A general solution of (*) is $X(x)=A\cos\lambda x + B\sin\lambda x$.

$X'(x) = -\lambda A\sin\lambda x + \lambda B\cos\lambda x$.

Satisfying $X'(0) = 1$, we get B=0 or λ=0. We consider these cases separately. For B=0 and satisfying $X'(1) = 0$, we get $\lambda A\sin\lambda$=0.

Solving the equation $\sin\lambda$=0, we get $\lambda_k = \pi k$. Hence, $X_k = \cos\pi k x$, $k = 1, 2, \ldots$.

If λ=0, $X_0 = const$. Finally, the system of functions is $X_k = \{1, \ \cos \pi kx\}$

For each $\lambda_k \neq 0$, the solution of the equation

$$T' + \lambda_k^2 T = 0 \quad (\lambda_k = \pi k)$$

is $T_k(t) = B_k \exp(-\lambda_k^2 t)$. If λ=0, $T_0 = const = B_0$

Therefore, the solution to the problem (1), (3) can be written as

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)\, X_k(x) = B_0 + \sum_{k=1}^{\infty} B_k \exp(-\pi^2 k^2 t) \cos(\pi k x)$$

(c) Then find the solution that satisfies (2) (Find coefficients B_k). (1p)

<u>Solution</u>

The function *u(x,t)* should satisfy the initial condition *u(x,0)=f(x)*.
From (4) and (2) we obtain:

$$u(x,0) = B_0 + \sum_{k=1}^{\infty} B_k \cos(\pi k x) = f(x) .$$

Therefore, B_k are coefficients of Fourier cosine series.

(d) Find the solution to the problem (1), (2), (3), if $f(x)=1+\cos 2\pi x$ (2p)

Solution.

The Fourier sine coefficients of $f(x)= 1+\cos 2\pi x$ are $B_0=1$, $B_2=1$ and $B_k=0$ for other *k*.
Therefore, the solution (4) to the problem is

$$u(x,t) = 1 + \exp(-4\pi^2 t) \cos(2\pi x) .$$