Aalto University, School of Science

Mat-1.3651 Matrix computations **Eirola/Murtola**

Partial exam 2, April 29, 2013

No calculators or texts are allowed. Time for the exam is three hours.

1. Measured in $\| \ \|_2$–norm, how far from

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & -2 \\ 2 & 0 & 0 \end{bmatrix}$$

is the closest singular matrix? Hint: SVD.

2. Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be nonzero and $\boldsymbol{x}_1 \in \mathbb{C}^n$ a unit vector such that

$$\|\boldsymbol{A}\boldsymbol{x}_1\| = \max_{\|\boldsymbol{x}\|_2=1} \|\boldsymbol{A}\,\boldsymbol{x}\|_2 \ ,$$

and set $\boldsymbol{x}_2 = \boldsymbol{A}\boldsymbol{x}_1 / \|\boldsymbol{A}\boldsymbol{x}_1\|$. Let $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ be Householder transformations such that $\boldsymbol{H}_j\,\boldsymbol{x}_j$, $j = 1, 2$ are parallel to $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$. Show that, except for the $\boldsymbol{B}_{1,1}$ element, all other entries in the first row and the first column of $\boldsymbol{B} = \boldsymbol{H}_2\,\boldsymbol{A}\,\boldsymbol{H}_1$ are zero. Hint: $\boldsymbol{H}_j = \boldsymbol{H}_j^* = \boldsymbol{H}_j^{-1}$.

3. Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ be Hermitian and tridiagonal with all $\boldsymbol{A}_{j,j+1}$, $j = 1, \ldots, n-1$ elements nonzero. Show that the eigenvalues of A are distinct. (Hint: look at the rank of $\boldsymbol{A} - \lambda\,\boldsymbol{I}$.)

   Give an example in the non-Hermitian case, where this is not true.

4. Let $\boldsymbol{A} \in \mathbb{C}^{n\times n}$, $\boldsymbol{b} \in \mathbb{C}^n$, and assume $\mathcal{K} = \operatorname{span}\{\boldsymbol{A}^j\boldsymbol{b} \mid j = 0, 1, \ldots\}$ has dimension $k \geq 1$. Show that the vectors $\boldsymbol{b}, \boldsymbol{A}\,\boldsymbol{b}, \ldots, \boldsymbol{A}^{k-1}\,\boldsymbol{b}$ are linearly independent. Let $\boldsymbol{P} = \boldsymbol{Q}_k\,\boldsymbol{Q}_k^*$ be the orthogonal projection onto $\mathcal{K}$ and set $\boldsymbol{B} = \boldsymbol{A}\,\boldsymbol{P}$. Show that $\Lambda(\boldsymbol{B}) \subset \Lambda(\boldsymbol{A}) \cup \{0\}$.

5. Assume $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ are symmetric and positive definite. Show that the 2-norm condition number satisfies

$$\kappa_2(\boldsymbol{A} + \boldsymbol{B}) \leq (\max \Lambda(\boldsymbol{A}) + \max \Lambda(\boldsymbol{B}))/(\min \Lambda(\boldsymbol{A}) + \min \Lambda(\boldsymbol{B})) \ .$$

5. Assume $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is diagonalizable and has $k \leq n$ distinct eigenvalues. Show that GMRes gives the solution in k steps.