Aalto University, School of Science

Mat-1.3651 Matrix computations **Eirola/Murtola**

Partial exam 1, March 9, 2013

No calculators or texts are allowed. Time for the exam is three hours.

1. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times p}$, and $\boldsymbol{C} \in \mathbb{R}^{p\times q}$ be given. We want to compute the product $\boldsymbol{ABC}$. This can be done either as
$$a) \qquad \boldsymbol{A}(\boldsymbol{BC}) \qquad \text{or} \qquad b) \qquad (\boldsymbol{AB})\boldsymbol{C}\ .$$
Compute the complextity (real (or floating point) number multiplications) for both cases. For which values of n, p, q the form a) is more favourable?

2. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $m > n$ and $\boldsymbol{A} = \boldsymbol{QR}$ its QR-decomposition with $\boldsymbol{Q} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{R} \in \mathbb{R}^{n\times n}$ Show that $\boldsymbol{R}$ is invertible if and only if the columns of $\boldsymbol{A}$ are linearly independent.

3. Assume matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar. Show that they have the same number of linearly independent eigenvectors.

4. Show that $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ is normal if and only if $\|\boldsymbol{A}\|_F^2 = \sum_{j=1}^n |\lambda_j|^2$, where $\|\boldsymbol{A}\|_F^2 = \sum_{i,j=1}^n |a_{i,j}|^2$ and λ_j's are the eigenvalues of $\boldsymbol{A}$.
Hint: the Schur form.

5. Let $\boldsymbol{A} = \begin{bmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{I} & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$, where $\boldsymbol{I} \in \mathbb{R}^{n\times n}$.
a) Compute the eigenvalues of $\boldsymbol{A}$. For this consider the equation
$$\boldsymbol{A} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = \lambda \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$$
and you will find only two eigenvalues.
b) Assume $\boldsymbol{E} \in \mathbb{R}^{n\times n}$ satisfies $\|\boldsymbol{E}\|_2 \le \frac{1}{2}$. Find the smallest possible set where the eigenvalues of
$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{E} \end{bmatrix}$$
are sure to be.