Mat-1.3650 Finite element method I
Easy exam, 6.5.2013

Remember to write your **name and student number** on every paper.
You will only need a pencil, and perhaps an eraser, to do this exam.
The time allowed is three hours.
You can answer in English or Finnish; also in Swedish if you must.

This is an exam: **Provide step by step answers and justify your reasoning.**

1. A) Let V be a Hilbert space with a norm $\|\cdot\|_V$. Suppose we have a bilinear form $a : V \times V \to \mathbb{R}$ and a linear functional $L : V \to \mathbb{R}$. Define the following

   - Coercive bilinear form
   - Bounded bilinear form
   - Bounded linear functional

   B) Define $L^2(\Omega)$- and $H^1(\Omega)$-norms.
   Which is true: $L^2(\Omega) \subset H^1(\Omega)$ or $H^1(\Omega) \subset L^2(\Omega)$? **Why so?**

2. A) Derive the weak form of the problem

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

   in which the domain $\Omega \subset \mathbb{R}^n$ is convex with a smooth boundary $\partial\Omega$ and $f \in L^2(\Omega)$. You can assume that you look for the solution in the space $H_0^1(\Omega)$.
   B) Show that the bilinear form is bounded and coercive in $H_0^1(\Omega)$.

3. Let V be a Hilbert space and suppose we have a bilinear form $a : V \times V \to \mathbb{R}$ and a linear functional $L : V \to \mathbb{R}$. Let $u \in V$ and $u_h \in V_h \subset V$ be such that

$$a(u, v) = L(v) \quad \forall v \in V \quad \text{and} \quad a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h.$$

   For $v \in V$, let $\|v\|_E^2 := a(v, v)$ denote the energy norm of the problem. Assuming it holds

$$|a(w, v)| \leq \|w\|_E \|v\|_E \quad \forall w, v \in V$$

   show that

$$\|u - u_h\|_E \leq \inf_{v_h \in V_h} \|u - v_h\|_E.$$

   Hint: Galerkin orthogonality.

4. Consider our model Poisson problem $-\Delta u = f$ with homogenous Dirichlet boundary conditions in a bounded domain Ω. Suppose we have the exact solution u and the finite dimensional solution $u_h \in V_h \subset H_0^1(\Omega)$. Suppose the following a priori estimate holds:

$$\|u - u_h\|_1 \leq Ch\|u\|_2.$$

   What does this estimate tell you of the convergence of the error? (What is h? What does C depend on? Etc...) Be verbal and show what you have learned.