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Mat-1.3460 Principles of Functional Analysis

1st mid-term exam, October 20th, 2012

Exam time 3 hours. Calculators are not allowed. You can write your answers in Finnish, Swedish or English.

1. (6p) Let $X := C[0,1]$ be a Banach space equipped with the norm $\|f\| = \max_{0\le t\le 1} |f(t)|$. Define a linear operator $A : C[0,1] \to C[0,1]$ by

$$(Af)(t) = \int_0^t e^{t-s} f(s)\, ds.$$

   (a) Show that $A \in B(X)$ and calculate the norm $\|A\|$.

   (b) It is easy to see that A is one-to-one (injection) using derivate (you don't have to prove this). But is A onto (surjection)? (Justify your answer.)

   (c) What can you say about the existence of an inverse of A? If A has an inverse, is it continuous?

2. (6p) Let X be a Banach space.

   (a) Define the dual X' and the bidual X'', and explain what is meant by the claim $X \subset X''$.

   (b) Define the concept reflexive space, and tell which l^p spaces are reflexive and what are their duals. (Proofs not required.)

3. (6p) Let $X = \mathbb{R}^2$. Assume given a nonempty compact set $K \subset \mathbb{R}^2$ and some norm $\|\cdot\|$ in the space $\mathbb{R}^2$.

   (a) Explain why for every $a \in \mathbb{R}^2$ there is a point in K closest to the point a.

   (b) Show that if the norm in X is the maximum-norm, $\|x\| = \max\{|x_1|, |x_2|\}$, and K is the unit ball of X, $K = \{x \in X : \|x\| \le 1\}$, then the closest point does not have to be unique.

   (c) It has been shown in the course that if K is a nonempty convex and closed subset of a Hilbert space, then for every given point there exists a unique closest point in K. Show directly without referring to this general fact, that if the norm in $\mathbb{R}^2$ is given by an inner product and K is convex and closed, then the closest point is unique.