Department of Mathematics and Systems Analysis
Aalto University, School of Science

Nevanlinna/Lehtelä

Mat-1.2990 Foundations of Modern Analysis

2^{nd} mid-term exam on May 4th 2013 at 10-13

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Degree Programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KTA, KON, MAR, MTE, PUU, RRT, TFM, TIK, TLT, TUO, YYT.

No calculators allowed. The exam will last 3 hours.

1. Let $C \subset [0,1]$ be the usual Cantor set (constructed by removing the middle thirds) and let λ be the Lebesgue measure.
   (a) Show that $\lambda(C) = 0$.
   (b) Define the sets $C_q \subset \mathbb{R}$ for each $q \in \mathbb{Q}$ by
   $$C_q := C + q = \{x + q \in \mathbb{R} : x \in C\}.$$
   Define
   $$E := \bigcup_{q\in\mathbb{Q}} C_q$$
   Compute $\lambda(E)$. Justify your answers!

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.
   (a) Define when a function $f : X \to \mathbb{R}$ is measurable (denoted by $f \in \mathrm{M}$).
   (b) Let $f \in \mathrm{M}$ and let $g : X \to \mathbb{R}$ be such that $g(x) = -|f(x)|^2$.
   Show that $g \in \mathrm{M}$.

3. Some important convergence theorems were presented during the course. One of them is useful for computing the integral
   $$\lim_{n\to\infty} \int_0^\infty \cos(x^N/n)\, e^{-x^2}\, \mathrm{d}x,$$
   where $N > 0$ is an integer.
   (a) Formulate the theorem.
   (b) Compute the integral using the theorem you formulated in the (a)-part.