Tfy-56.4425 Fundamentals of Plasma Physics for Space and Fusion Applications Exam 22.5.2013

Answer to **FOUR** questions.

Plasma phenomena and general aspects

1. Give short (not more than 1/3 page for each) but complete answers to following questions:
   a) Explain (use pictures) why the electric potential introduced by a point charge in a plasma decays exponentially faster compared to the case if the point charge was placed in the vacuum. (1p)
   b) Why both poloidal and toroidal magnetic field is needed to confine the tokamak plasma? You can use either fluid or single particle picture in your explanation. (1p)
   c) The interplanetary magnetic field is an outward extension of the solar magnetic field. Why does it have a spiral-like shape? (1p)
   d) Within linearized kinetic theory, we observed either exponential damping or growth of plasma waves. As an exercise you calculated this exponential damping for Ion-acoustic wave. In reality, the waves, however, do not necessary continue the exponential growth forever. Why? (1p)
   e) List similarities/differences of Alfvén waves in MHD, cold plasma theory and kinetic theory. (1p)
   f) Define stability of the MHD plasma. What is the difference between the partial time derivative ($\frac{\partial}{\partial t}$) and total time derivative ($\frac{d}{dt}$) in MHD theory? (1p)

Single particle dynamics

2. a) Starting from the charged particle Lagrange function

$$L = \frac{1}{2} m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + q\dot{\boldsymbol{r}} \cdot \boldsymbol{A}(\boldsymbol{r}, t) - q\Phi(\boldsymbol{r}, t),$$

   derive the equation of motion for $\boldsymbol{r}$ using the Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}}$. (4p)

   b) Then calculate the time rate of change of the total energy

$$H = \frac{1}{2} m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + q\Phi(\boldsymbol{r}, t),$$

   and show that for time-independent electromagnetic potentials, the total energy, H, is conserved. (2p)

Continuity and momentum equations

3. Recall the statistical definitions for scalar density, velocity vector, and pressure tensor

$$n_s(\boldsymbol{r}, t) = \int f_s(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{v}, \quad \boldsymbol{V}_s(\boldsymbol{r}, t) = \frac{1}{n_s} \int \boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{v}$$

$$\boldsymbol{p}_s(\boldsymbol{r}, t) = \int m_s \left[\boldsymbol{v} - \boldsymbol{V}_s(\boldsymbol{r}, t)\right] \left[\boldsymbol{v} - \boldsymbol{V}_s(\boldsymbol{r}, t)\right] f_s(\boldsymbol{r}, \boldsymbol{v}, t) d\boldsymbol{v}.$$

   where the distribution function, f_s, satisfies the kinetic equation

$$\frac{\partial f_s}{\partial t}(\boldsymbol{r}, \boldsymbol{v}, t) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f_s(\boldsymbol{r}, \boldsymbol{v}, t) + \frac{q_s}{m_s} \left(\boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}, t)\right) \cdot \nabla_{\boldsymbol{v}} f_s(\boldsymbol{r}, \boldsymbol{v}, t) = C_s[f](\boldsymbol{r}, \boldsymbol{v}, t).$$

   Using the expressions above, and the assumption that the distribution function vanishes at the phase-space boundary ($|\boldsymbol{v}| \to \infty$),

   a) derive the continuity equation

$$\frac{\partial}{\partial t} n_s + \nabla_{\boldsymbol{r}} \cdot (n_s \boldsymbol{V}_s) = 0$$

   taking the zeroth moment of the kinetic equation (multiply by unity and integrate over the velocity space). Remember that the collision conserve particles ($\int C_s[f] d\boldsymbol{v} = 0$). (2p)

b) Also, derive the momentum equation

$$m_s n_s \left(\frac{\partial}{\partial t} \boldsymbol{V}_s + \boldsymbol{V}_s \cdot \nabla_r \boldsymbol{V}_s \right) + \nabla_r \cdot \boldsymbol{p}_s - q_s n_s \left(\boldsymbol{E} + \boldsymbol{V}_s \times \boldsymbol{B} \right) = \boldsymbol{F}_s,$$

taking the first moment of the kinetic equation. (multiply by $m\boldsymbol{v}$ and integrate over the velocity space). To get the desired result you will have to apply the continuity equation during your calculations. Remember that the collisional force density is $\boldsymbol{F}_s = \int m_s \boldsymbol{v} C_s[f] d\boldsymbol{v}$. (4p)

MHD stability

4. Consider a cylindrically symmetric plasma column ($\partial_z = 0, \partial_\theta = 0$; z is the direction of the cylinder axis) under equilibrium conditions, confined by a magnetic field. Verify that in cylindrical coordinates the hydromagnetic equilibrium is given by

$$\frac{dp(r)}{dr} = j_\theta(r) B_z(r) - j_z(r) B_\theta(r). \text{ (2p)}$$

Using Ampere's law, derive the equilibrium equation to the form

$$\frac{d}{dr} \left(p + \frac{B_\theta^2}{2\mu_0} + \frac{B_z^2}{2\mu_0} \right) = -\frac{B_\theta^2}{\mu_0 r}. \text{ (3p)}$$

Give a physical interprepation why the two terms including magnetic field are organized to the left hand side and one to the right hand side. (1p)

Linearization

5. During the course we studied linear phenomena by linearizing the set of equations. Now, you need to linearize and re-organize the following MHD equations

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{V} \times \boldsymbol{B})$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$

$$\rho \frac{\partial \boldsymbol{V}}{\partial t} + \rho (\boldsymbol{V} \cdot \nabla) \boldsymbol{V} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0$$

to arrive at a single equation for the perturbed plasma fluid velocity:

$$\left[\omega^2 - \frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)^2}{\mu_0 \rho_0} \right] \boldsymbol{V}_1 =$$

$$\left\{ \left[\frac{\gamma p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0} \right] \boldsymbol{k} - \frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)}{\mu_0 \rho_0} \boldsymbol{B}_0 \right\} (\boldsymbol{k} \cdot \boldsymbol{V}_1)$$

$$- \frac{(\boldsymbol{k} \cdot \boldsymbol{B}_0)(\boldsymbol{V}_1 \cdot \boldsymbol{B}_0)}{\mu_0 \rho_0} \boldsymbol{k}. \text{ (6p)}$$