Answer all four problems, which are graded on a scale 0-6 points.

Problem 1

Are the following statements true of false? Why?

(a) A hyperplane is a polyhedron.
(b) The basic solution $x_B = (x_{B(1)}, \ldots, x_{B(m)}) > 0$ is degenerate.
(c) If phase I of the two phase Simplex has optimal cost 0, then the problem in phase II is feasible.
(d) If the flow conservation constraints $Af = b$ of flow vector $f \geq 0$ are such that $\sum_i b_i \neq 0$, then f cannot be feasible.

Answer:

(a) True. By definition, a polyhedron is a set of real vectors that can be described using linear inequalities. A hyperplane is $\{x : \mathbb{R}^n | a^T x \leq b, -a^T x \leq -b\}$.

(b) False. All degenerate solutions have at least one basic variable at zero level. This is because in x_B there are m active constraints in $Ax = b$ and $n - m$ active non-negativity constraints among the non-basic variables. For degeneracy, we need more than n active constraints. Then at least one of the constraints $x_B \geq 0$ must be active (and hence $x_i = 0$ for some $i \in \{B(1), \ldots, B(m)\}$), wherefore the basic solution $x_B > 0$ cannot be degenerate.

(c) True. In phase I, the objective value is the sum of the (non-negative) deviations from the constraints, and if this value use zero, the first phase has identified a solution that does not conflict with any constraints and hence the problem for the second phase is feasible.

(d) True. If $\sum_i b_i \neq 1$, then either the capacities of the sources exceed the sinks or vice versa, wherefore flow should either be created from or vanish to nothing, which is prevented by the flow conservation constraints.

Problem 2

A company invests in projects $j = 1, \ldots, 3$. All projects cannot be launched within the budget, wherefore the company has formulated the following mixed integer linear program for selecting the projects.

$$\max \quad \sum_{j=1}^{3} r_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{3} c_{jt} x_j \leq b_t, \quad t = 1, 2, 3$$

$$x_j \in \{0, 1\}, \quad j = 1, 2, 3 \ ,$$

where r_j is the profit of project j, c_{jt} is the cost of project j in year t and b_t is the budget of year t. The constant parameters b_t, r_j and c_{jt} can be found from the table below.

(a)   Solve this problem's LP-relaxation with full tableau-simplex starting from $(x_1, x_2, x_3) =$
       $(0, 0, 0)$. Use Bland's rule. (3 p)

(b)   Show that $x_1 + x_2 + 2x_3 \leq 2$ is a cutting plane. (1 p)

(c)   Add the cutting plane to the constraints of the relaxation, form a dual feasible
       initial tableau for dual simplex and do one iteration (you do not have to compute
       the basic directions). Have you found the optimal solution of the original mixed
       integer problem? (2 p)

Hint: When adding a inequality of the form $\mathbf{a}'_{m+1}\mathbf{x} \geq b_{m+1}$, then the new basic directions
are given by

$$\bar{\mathbf{B}}^{-1}\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{A} & \mathbf{0} \\ \mathbf{a}'\mathbf{B}^{-1}\mathbf{A} - \mathbf{a}'_{m+1} & 1 \end{bmatrix} ,$$

where $\mathbf{a} = \big((\mathbf{a}_{m+1})_{B(1)}, \ldots, (\mathbf{a}_{m+1})_{B(m)}\big)$ and $B(1), \ldots, B(m)$ are the basic indices.

Cost	Year 1	Year 2	Year 3	Profit
Project 1	2	1	1	20
Project 2	4	7	4	40
Project 3	3	9	2	20
Budget	8	15	6	

Answer:

(a)   The binary constraints $x_i \in \{0, 1\}$ are replaced by constraints $x_i \geq 0$ and $x_i \leq 1$,
       and slack variables $s_i, i = 1 \ldots, 6$ are introduced to other except non-negativity
       constraints. Then the full tableau simplex can be iterated as follows:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	s_6
	-20	-40	-20						
8	2	4	3	1					
15	1	7	9		1				
6	1	4	2			1			
1	1						1		
1		1						1	
1			1						1
20		-40	-20				20		
6		4	3	1			-2		
14		7	9		1		-1		
5		4	2			1	-1		
1	1						1		
1		1						1	
1			1						1
60			-20				20	40	
2			3	1			-2	-4	
7			9		1		-1	-7	
1			2			1	-1	-4	
1	1						1		
1		1						1	
1			1						1
70						10	10		
0.5				1		-1.5	-0.5	2	
2.5					1	-4.5	3.5	11	
0.5			1			0.5	-0.5	-2	
1	1						1		
1		1						1	
0.5						-0.5	0.5	2	1

(b) The new constraint is equivalent to the third year budget constraint when the solutions are binary, because only the solution $(x_1, x_2, x_3) = (1, 1, 1)$ is prevented by both constraints. However, this constraint is violated by the current solution, because $1 + 1 + 2 \cdot \frac{1}{2} = 3 > 2$, wherefore it is a cutting plane.

(c) We introduce the new constraint $-x_1 - x_2 - 2x_3 - s_7 = 2$ with $s_7 \geq 0$. Then the current base is infeasible but it is dual feasible and can be used as the initial tableau for the dual simplex:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	s_6	s_7
70						10	10			
0.5				1		−1.5	−0.5	2		
2.5					1	−4.5	3.5	11		
0.5			1			0.5	−0.5	−2		
1	1						1			
1		1						1		
0.5						−0.5	0.5	2	1	
−1						−1		3		1
60							10	30		10
2				1			−0.5	−2.5		−1.5
7					1		3.5	−2.5		−4.5
0			1				−0.5	−0.5		
1	1						1			
1		1						1		
0							0.5	0.5	1	−0.5
1						1		−3		−1

This table is optimal with objective value 60, and because the variables $x_1 = 1, x_2 = 1$ and $x_3 = 0$ are all integer, this is the solution of the integer programming problem as well.

Problem 3

You are making a schedule for a high school class over one period. Courses are held in time slots 8-10, 10-12,12-14 or 14-16. During the period, the class has n courses that can be held on Tuesday or Wednesday; m courses that can be held on Monday, Tuesday or Friday; and the remaining k courses can be held on any weekday from Monday to Friday. Each course has teaching in exactly one time slot and one weekday. The class can have only one course at the same time. Also, on two consecutive days, there cannot be teaching in all the time slots of theses days. Formulate a mixed integer linear program that yields a schedule where the school day that has the most courses during the week will has few courses as possible. Explain how your modeling techniques work. *Hint:* An indicator variable of the form x_{ijk} can be helpful.

Answers:

Let

$$x_{ijk} = \begin{cases} 1, & \text{if class } i \text{ is held on weekday } j \text{ in time slot } k , \\ 0, & \text{otherwise}, \end{cases}$$

where $i \in I = \{1, \ldots, (n + m + k)\}$, $j \in J = \{1, \ldots, 5\}$ and $k \in K = \{1, 2, 3, 4\}$. Then

the problem can be formulated as follows:

$$\min z \tag{1}$$

$$\text{s.t} \sum_{j \in \{2,3\}, k \in K} x_{ijk} = 1 \forall i \in \{1, \ldots, n\} \tag{2}$$

$$\sum_{j \in \{1,2,5\}, k \in K} x_{ijk} = 1 \forall i \in \{n+1, \ldots, n+m\} \tag{3}$$

$$\sum_{j \in J, k \in K} x_{ijk} = 1 \forall i \in \{n+m+1, \ldots, n+m+k\} \tag{4}$$

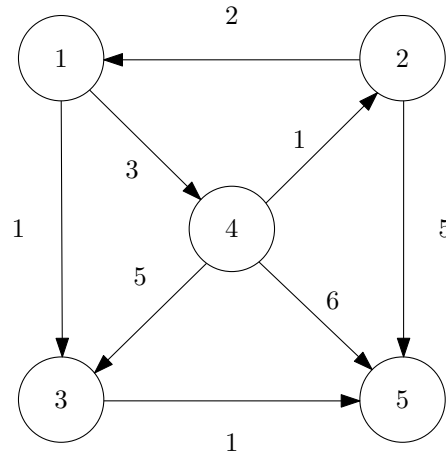$$\sum_{i \in I} x_{ijk} \leq 1 \forall j \in J, k \in K \tag{5}$$

$$\sum_{i \in I, k \in K} x_{i,j-1,k} \geq \sum_{i \in I, k \in K} x_{ijk} + 1 \forall j \in \{2, \ldots, 5\} \tag{6}$$

$$z \geq \sum_{i \in I, k \in K} x_{ijk} \forall j \in J \tag{7}$$

$$x_{ijk} \in \{0, 1\} \forall i \in I, j \in J, k \in K . \tag{8}$$

Problem 4

Consider the network below in which next to each arc is the length of that arc.

(a) Find the shortest path to node 5 from the other nodes using Dijkstra's algorithm. (3 p)

(b) Draw a minimum cost network flow problem that corresponds to the problem in part (a), and find its solution. *Hint:* Add unit source to nodes 1-4 and a sink in node 5 with capacity 4. (2 p)

(c) Determine the optimal dual variables and interpret them. (1 p)

Answers:

(a) Iteration 1: $\ell = 3$, $p_3^* = 1$, $c_{15} = 2$ (new arc), remove node 3.
Iteration 2: $\ell = 1$, $p_1^* = 2$, $c_{25} = 4$ (improved arc), remove node 1.
Iteration 3: $\ell = 2$, $p_2^* = 4$, $c_{42} = 5$ (improved arc), remove node 2.

Iteration 4: $\ell = 4$, $p_4^* = 5$, algorithm is complete.

The shortest path is on the path 42135.

(b) Each nodes has a unit source except the destination node 5, which has a sink with capacity 4. The arc lengths correspond to flow costs, wherefore carrying one unit of flow along an arc will increase the objective value by the length of the arc. The corresponding graph is below and the optimal basis is indicated by the dashed arcs. The flow corresponding to this basis is $f_{42} = 1$, $f_{21} = 2$, $f_{13} = 3$, $f_{35} = 4$, others zero.
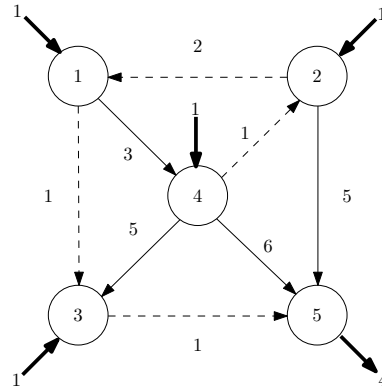
(c) The optimal dual variables are $p_1 = 2$, $p_2 = 4$, $p_3 = 1$, $p_4 = 5$ and $p_5 = 0$, where p_i can be interpreted as the optimal route length from node i to node 5.