Vastaa kaikkin neljään tehtävään, jotka kukin arvostellaan asteikolla 0-6 pistettä.

Tehtävä 1

Ovatko seuraavat väittämät oikein vai väärin? Perustele vastauksesi.

(a) Lineaarisen kokonaislukutehtävän käypä alue on monitahokas.

(b) Puuratkaisu määrittää käyvän virtauksen verkossa.

(c) Kokonaislukutehtävän konveksi kuori vastaa sen LP-relaksaation käypää aluetta.

(d) Kun simplex-algoritmissa askelpituus $\theta > 0$, siirrytään ratkaisuun, jossa kohdefunktion arvo on aidosti parempi.

Tehtävä 2

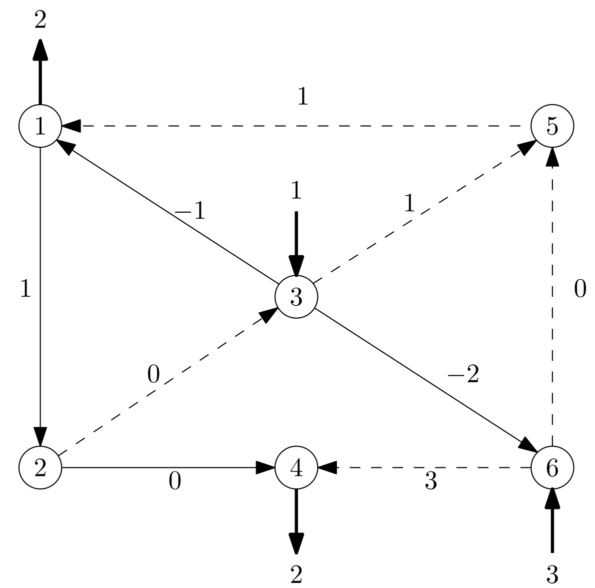
Pallo sijaitsee pisteessä $x_0 = (5, -2, 3)$. Siihen vaikuttaa painovoima, joka on vektorin $(0, 0, -1)$ suuntainen. Pallo ei voi läpäistä sen liikettä rajoittavia tasoja $x_1 + 2x_3 = 2$, $-x_1 - 3x_2 = 1$, $x_2 + 2x_3 = 1$, $-x_1 + x_2 + 2x_3 = -1$, $x_1 = 0$ ja $x_2 = 0$.

(a) Muodosta lineaarisen ohjelmoinnin tehtävä, joka ratkaisee pallon sijainnin kun se on päätynyt lepoon. (2 p)

(b) Osoita, että $x^* = (2, -1, 1)$ on muodostetun tehtävän optimaalinen ratkaisu. (4 p)

Tehtävä 3

Ratkaise oheinen kapasiteettirajoittamaton minimivirtaustehtävä verkkosimplexillä. Aloita katkoviivoin merkitystä ratkaisusta. Kunkin kaaren vieressä on sen kustannus. Lähteitä /nieluja merkitään nuolilla, joiden päässä lukee lähteen/nielun kapasiteetti.



Tehtävä 4

Tarkastellaan tehtävää

$$\begin{aligned}
\max \quad & 51E + 102C + 66P_1 + 66P_2 + 89B \\
\text{s.t.} \quad & 10E + 15C + 10P_1 + 10P_2 + 20B \leq 130 \quad \text{(Clay)} \\
& E + 2C + 2P_1 + P_2 + B \leq 13 \quad \text{(Enamel)} \\
& 3E + C + 6P_1 + 6P_2 + 3B \leq 45 \quad \text{(Dry Room)} \\
& 2E + 4C + 2P_1 + 5P_2 + 3B \leq 23 \quad \text{(Kiln)} \\
& P_1 - P_2 = 0 \quad \text{(Primrose)} \\
& E, C, P_1, P_2, B \geq 0 \ ,
\end{aligned}$$

ja sen optimaaliseen ratkaisuun liittyviä herkkyysanalyysitaulukkoita (ohessa).

(a) Tulkitse muuttujan E redusoitu kustannus. Kerro tulkintasi taustaoletukset.

(b) Miten optimaalinen kustannus muuttuu, jos rajoitteen Kiln oikeaa puolta muutetaan arvosta 23 arvoon 20? Entä jos muuttujan C kerrointa kohdefunktiossa muutetaan arvosta 102 arvoon 95?

(c) Kuinka paljon voi muuttujan P_2 kerrointa rajoitteessa Clay muuttaa ilman, että optimaalinen kanta vaihtuu?

(d) Onko taulukon primaaliratkaisu degeneroitunut? Entä duaaliratkaisu?

Variable	Optimal Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
E	0	-3.571	51	3.571	∞
C	2	0	102	16.667	12.5
P_1	0	0	66	37.571	∞
P_2	0	-37.571	66	37.571	∞
B	5	0	89	47	12.5

Constraint Name	Slack Value	Dual Variable	Constraint RHS	Allowable Increase	Allowable Decrease
Clay	130	1.429	130	23.33	43.75
Enamel	9	0	13	∞	4
Dry Room	17	0	45	∞	28
Kiln	23	20.143	23	5.60	3.50
Primrose	0	11.429	0	3.50	0

Answers

Problem 1

(a) False. All polyhedrons are convex sets, but for instance the valid mixed integer linear programming constraint $x \in \{0,1\}$ defines a non-convex set, because the convex combination $\lambda \cdot 0 + (1-\lambda) \cdot 1$ does not belong to the feasible set when $0 < \lambda < 1$.

(b) False. A tree solution defines a basic solution, but it does not need to be a basic feasible solution.

(c) False. The convex hull of the set $\{x \in \mathbb{N} | \frac{1}{2} \leq x \leq 1\}$ is $\{1\}$, whereas the LP-relaxation yields the interval $[\frac{1}{2}, 1]$.

(d) True. The simplex algorithm moves only to directions that improve the objective value, wherefore by taking a positive step $\theta > 0$ into such a direction will improve the objective value. (if $\theta = 0$, then the pivot is degenerate).

Problem 2

(a) By substituting the initial point $x_0 = (5, -2, 3)$ into the equations of the constraining planes, we get the area within which the ball is contained. For instance, at x_0, the left hand side of the first plane $x_1 + 2x_3 = 2$ is $5 + 2 \cdot 3 = 11 \geq 2$, wherefore the first plane constrains the ball to lie in the halfspace $x_1 + 2x_3 \geq 2$. The ball will roll as low as possible, wherefore the objective is to minimize x_3. This gives the problem

$$\begin{aligned} \min \quad & x_3 \\ \text{s.t } x_1 + 2x_3 &\geq 2 \\ -x_1 - 3x_2 &= 1 \\ x_2 + 2x_3 &\geq 1 \\ -x_1 + x_2 + 2x_3 &= -1 \\ x_1 &\geq 0 \\ x_2 &\leq 0 \\ x_3 & \text{ free} \end{aligned}$$

(b) We prove that $x^* = (2, -1, 1)$ is the optimal solution of the problem above using duality. The dual of the problem is

$$\begin{aligned} \max \quad & 2p_1 + p_2 + p_3 - p_4 \\ \text{s.t } p_1 - p_2 - p_4 &\leq 0 \\ -3p_2 + p_3 + p_4 &\geq 0 \\ 2p_1 + 2p_3 + 2p_4 &= 1 \\ p_1, p_3 &\geq 0 \\ p_2, p_4 & \text{ free} \end{aligned}$$

Using the complementary slackness condition, we see that all constraints in the dual are active (because $x_1, x_2, x_3 \neq 0$) and because the first primal constraint is not active, then $p_1 = 0$. Now let us solve for the remaining p_2, p_3, p_4:

$$\begin{aligned} -p_2 - p_4 &\leq 0 \\ -3p_2 + p_3 + p_4 &\geq 0 \\ 2p_3 + 2p_4 &= 1 \end{aligned}$$

which is solved by $p_2 = \frac{1}{6}, p_3 = \frac{2}{3}, p_4 = -\frac{1}{6}$. This solution fulfills also the constraints on the signs of the variables, wherefore it is a dual feasible solution. This solution has cost $2 \cdot 0 + \frac{1}{6} + \frac{2}{3} - (-\frac{1}{6}) = 1$, which is equal to the primal cost. Thus we have found a pair of feasible solutions to the primal and dual problems with the same cost, wherefore by strong duality, these solutions are optimal.

Problem 3

The initial flow in the basic arcs are $f_{64} = 2, f_{65} = 1, f_{23} = 0, f_{35} = 1, f_{51} = 2$, which is feasible. The reduced cost of arc $(3, 1)$ is $-1 - 1 - 1 = -3 < 0$, wherefore we bring it to the basis. The flow in the resulting cycle can be increased by one unit, and arc $(3, 5)$ exits the basis. The new basic flows are $f_{64} = 2, f_{65} = 1, f_{23} = 0, f_{31} = 1, f_{51} = 1$.

The reduced cost of arc $(2, 4)$ is $0 + (-3) + 0 + 1 - (-1) - 0 = -1 < 0$, wherefore we bring it to the basis. The flow in the resulting cycle can be increased by zero units (degenerate pivot) and arc $(2, 3)$ exits the basis. The new basic flows are $f_{64} = 2, f_{65} = 1, f_{24} = 0, f_{31} = 1, f_{51} = 1$.

The reduced cost of arc $(1, 2)$ is $1 + 0 - 3 + 0 + 1 = -1 < 0$, wherefore we bring it to the basis. The flow of the resulting cycle can be increase two units and arc $(6, 4)$ exits the basis. The new basic flows are $f_{12} = 2, f_{65} = 3, f_{24} = 2, f_{31} = 1, f_{51} = 3$.

The reduced cost of arc $(2, 3)$ is $0 + 1 + 1 + 1 = 3 > 0$. The reduced cost of arc $(3, 5)$ is $1 + 1 - (-1) = 3 > 0$. The reduced cost of arc $(3, 6)$ is $-2 + 0 + 1 - (-1) = 0$. The reduced cost of arc $(6, 4)$ is $3 - 0 - 1 - 1 - 0 = 1 > 0$. Thus the reduced costs of all arcs are non-negative, and the algorithm stops. The optimal flow is $f_{12} = 2, f_{65} = 3, f_{24} = 2, f_{31} = 1, f_{51} = 3$ (others zero) with an optimal cost $2 \cdot 1 + 3 \cdot 0 + 2 \cdot 0 + 1 \cdot (-1) + 3 \cdot 1 = 4$.

Problem 4

(a) The reduced cost of variable E is -3.571. This means that increasing one unit of variable E such that the basic variables are adjusted to fulfill the constraints, will decrease the objective value by -3.571, that is the reduced cost. This interpretation is valid as long as the basic variables can be changed such that the solution remains feasible.

(b) The change in Kiln is within the allowable decrease. Thus the optimal objective value will change by $(20 - 23) \cdot 20.143 = -60.429$.

The change in the objective coefficient is also within allowable decrease. Thus the optimal objective value will change by $(95 - 102) \cdot 2 = -14$.

(c) P_2 corresponds to a non-basic column in the constraint matrix. Thus, the feasibility is not affected, but the reduced cost of P_2 is affected. For a non-basic column, we have

$$c_j - p^T(A_j + \delta e_i) \leq 0 \Rightarrow \bar{c}_j - \delta p_i \leq 0 \ .$$

Note that for a maximization problem, the reduced costs are non-positive at optimum. This yields the constraint $\delta \geq \frac{\bar{c}_j}{p_i} = \frac{-37.571}{1.429} = -26.2918$. Thus the coefficient can be decreased from its original value 10 to -16.2918 or increased infinitely without changing the basis.

(d) The primal solution is degenerate, because a basic variable (here P_2) is at zero level. Also the dual is degenerate, because there exists a nonbasic variable (here P_1) with zero reduced cost.