Tik–106.4100 Design and Analysis of Algorithms, autumn 2011

Exam, September 5th, 2012 **No calculators or extra material allowed**

Write the following clearly on top of each paper you submit: "T-106.4100 Design and Analysis of Algorithms, Sep 5th, 2012", your full name, student ID and study programme, and the total number of papers you submit.

1. a) (3p) Which of the following conjectures are correct and which are incorrect? Give a short justification for each answer (exact mathematical proofs are not required)!
   i. $3n^3 + 5n \log n \in \Omega(n^2)$
   ii. $3n^2 + 2n \in O(n^3)$
   iii. $5n^2 + 5n \log n \in \Theta(n^3)$

   b) (3p) Explain what amortized complexity means and how it can be analysed by using the accounting method or the potential method. The recommended length of your aswer is one page.

2. a) (3p) Solve the following recurrence, when n is a power of five. An exact answer is required (an answer in Θ or O notation is not enough).

$$T(n) = \begin{cases} 1, & \text{when } n = 1 \\ 2T(n/5) + 3n & \text{when } n > 1 \end{cases}$$

   b) (3p) Make a good guess to solve the following recurrence and check your result by using induction (c_1 and c_2 are constants).

$$T(n) \leq \begin{cases} c_1, & \text{when } n = 1 \\ 3T(n-1) + c_2 & \text{when } n > 1 \end{cases}$$

3. a) (2p) Consider disjoint-set forests supporting FIND–SET and UNION operations. Explain briefly, for which purposes these structures are used. Give an example of an algorithm which uses these structures and tell for which purposes the algorithm uses these structures. You do not have to explain the details of the algorithm itself.

   b) (4p) Give two Fibonacci-heap operations whose amortized time complexity is different from their worst case time complexity. Give the time complexities of these operations. Give also short justifications for the complexities (exact mathematical analysis is not required).

4. (6p) You are going on a long trip. You start on the road at kilometer post 0. Along the way there are n hotels, at kilometer posts $a_1, a_2, \ldots, a_n$, where each a_i is measured from the starting point and $a_1 < a_2 < \ldots < a_n$. The only places you are allowed to stop are at these hotels, but you can choose which of the hotels you stop at. You must stop at the final hotel (at distance a_n), which is your destination.

   You would ideally like to travel 300 kilometers a day, but this may not be possible (depending on the spacing of the hotels). If you travel x kilometers during a day, the penalty for that day is $(300 - x)^2$. You want to plan your trip so as to minimze the total penalty – that is, the sum, over all travel days, of the daily penalties. Give a dynamic programming algorithm that determines the optimal sequence of hotels at which to stop.

   Do not write the code of the algorithm, but present the expressions needed by the dynamic programming and tell how and in which order the values of these expressions are calculated. Explain also how the hotels where you should stay can be found out.

   If your algorithm is not based on dynamic programming, you cannot obtain full points from your solution.

5. (6p) A bipartite graph is a graph $G = (V, E)$ whose vertices can be partioned into two sets V_1 ja V_2 such that there are no edges between vertices in the same set (for instance, if $u \in V_1$ and $v \in V_1$, then there is no edge between u and v). Write a pseudocode for an algorithm which determines whether an undirected graph is bipartite. Your algorithm should run in $O(|V| + |E|)$ time.

   Hint: An undirected graph is bipartite if and only if it can be colored with just two colors such that for every edge $(u, v) \in E$, the color of vertex u is different from the color of vertex v.