3. Show that for two particle operators of the form

$$F = \frac{1}{2}\sum_{\alpha\neq\beta} f^{(2)}(\mathbf{x}_\alpha, \mathbf{x}_\beta) \tag{1}$$

can be written in the form

$$F = \frac{1}{2}\sum_{i,j,k,m} \langle i,j|f^{(2)}|k,m\rangle a_i^\dagger a_j^\dagger a_m a_k, \tag{2}$$

where

$$\langle i,j|f^{(2)}|k,m\rangle = \int d\mathbf{x}\int d\mathbf{y}\varphi_i^*(\mathbf{x})\varphi_j^*(\mathbf{y})f^{(2)}(\mathbf{x},\mathbf{y})\varphi_k(\mathbf{x})\varphi_m(\mathbf{y}). \tag{3}$$

Tips: Remember that $\sum_\alpha |i\rangle_{\alpha\alpha}\langle k| = a_i^\dagger a_k$. You may assume that particles are either bosons or fermions in which case you might also find respective commutation relations $[a_k, a_j^\dagger]_\mp = \delta_{kj}$ useful (upper sign for bosons). Also, the symmetry of $f^{(2)}$ implies that matrix elements are independent of α and β.

4. A coherent state of a bosonic many-body system is given by

$$|\Psi\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle_{\text{Fock}},$$

where $|n\rangle_{\text{Fock}}$ is the Fock state in which n particles have been created in some single-particle state i (let us denote that single-particle state $i=0$), i.e. $|n\rangle_{\text{Fock}} = \mathcal{N}\left(c_0^\dagger\right)^n|0\rangle$, where $|0\rangle$ is the vacuum state with zero particles.

a) Find a proper normalization $\mathcal{N}$ for the Fock state $|n\rangle_{\text{Fock}}$ and calculate the norm of the coherent state $\langle\Psi|\Psi\rangle$.
b) Find the number of particles in the coherent state $\langle\Psi|c_0^\dagger c_0|\Psi\rangle$.
c) Find the variance of the number of particles in the coherent state $\langle\Psi|c_0^\dagger c_0 c_0^\dagger c_0|\Psi\rangle - \left(\langle\Psi|c_0^\dagger c_0|\Psi\rangle\right)^2$.