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Mat-1.2990 Foundations of Modern Analysis

Exam, 27.05.2013

1. Let $\mathbb{A}_2$ be the set consisting of the real roots of nonzero polynomials with integer coefficients of degree at most 2 i.e. $\alpha \in \mathbb{A}_2$ if $\alpha \in \mathbb{R}$ and there exist $a_0, a_1, a_2 \in \mathbb{Z}$ ($a_i \neq 0$ for some $i \in \{0,1,2\}$) such that

$$p(\alpha) = a_2\alpha^2 + a_1\alpha + a_0 = 0.$$

   (a) Show that $\mathbb{A}_2$ is countable.

   (b) Is the power set $\mathcal{P}(\mathbb{A}_2)$ countable? Justify your answers.

2. Let (X, d) be a metric space and let $K \subset X$ be compact.

   (a) What is the definition of compactness of a set K?

   (b) Let $A \subset X$ be a finite set. Prove using the definition of compactness, that $K \cup A$ is compact.

3. Let $\Lambda \in \mathcal{D}'$ be a distribution and let $(\Lambda_j)_{j=1}^{\infty}$ be a sequence of distributions ($\Lambda_j \in \mathcal{D}'$ for every j).

   (a) Give a definition for the distributional derivative $D\Lambda$.

   (b) Define convergence in the sense of distributions (denoted by $\Lambda_j \overset{\mathcal{D}'}{\to} \Lambda$).

   (c) Prove that if $\Lambda_j \overset{\mathcal{D}'}{\to} \Lambda$, then $D\Lambda_j \overset{\mathcal{D}'}{\to} D\Lambda$.

4. Let (X, d, μ) be a metric measure space, where μ is a Borel measure. Consider functions $f : X \to \mathbb{R}$.

   (a) Define when a function f is measurable (denoted by $f \in \mathrm{M}$).

   (b) Show that if f is continuous, then $f \in \mathrm{M}$.