Aalto University, Department of Mathematics and Systems Analysis.

Mat-1.3350, Partial Differential Equations L, fall 2013.

FIRST MIDTERM EXAM, OCTOBER 1ST, 2013

1. *Answer "yes" or "no", according to whether the following statements are correct, or false, respectively. In any case, briefly justify your answer.*

Note that in (ii) and (iii), $\hat{f} = \hat{f}(j)$, $j \in \mathbb{Z}$, *denotes the* Fourier coefficients *of* 2π*-periodic functions. In (iv) and (v),* $\hat{f} = \hat{f}(\xi)$, $\xi \in \mathbb{R}$, *denotes the* Fourier transform *of* f *on* $\mathbb{R}$.

(i) *Let* f *be an odd* 2π*-periodic function. Then the (real form of the) Fourier series of* f *has the form*

$$f \sim \sum_{n=1}^{\infty} c_n \sin(nt), \quad \text{for some constants } c_n, \quad n \in \mathbb{Z}.$$

(ii) *There exists a* 2π*-periodic function* f *satisfying*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|dx = 1 \quad \text{and} \quad \hat{f}(100) = 10.$$

(iii) *There exists a function* $f \in L^1([-\pi,\pi))$ *with Fourier coefficients* $\hat{f}(j) = j$ *for all* $j \in \mathbb{Z}$.

(iv) *There exists a function* $f \in L^1(\mathbb{R})$ *such that* $\hat{f}(\xi) = f(\xi)$ *for all* $\xi \in \mathbb{R}$.

(v) *There exists a function* $f \in L^1(\mathbb{R})$ *such that*

$$\hat{f}(\xi) = \begin{cases} \frac{1}{|\xi|^2}, & \text{if } |\xi| \geq 1, \\ 0 & \text{if } |\xi| < 1. \end{cases}$$

2. *Let* $f(x) = \sin x + \cos x + \sin 3x, \quad -\pi \leq x \leq \pi.$

(i) *Find the Fourier coefficients of* f, $\hat{f}(j)$, *for every* $j \in \mathbb{Z}$.

(ii) *Solve the steady state heat equation (that is, the Laplace equation) in the unit disc, with boundary data* f:

$$\begin{cases} \Delta u = 0 & \text{when} \quad x \in \mathbb{R}^2, |x| < 1, \\ u(x) = f(x) & \text{when} \quad x \in \mathbb{R}^2, |x| = 1. \end{cases}$$

3. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be a* C^2*-function with compact support (*f *is 0 outside an interval of the form* $[-R, R]$ *for some* $R > 0$*). We denote by* $\hat{f}(\xi)$, $\xi \in \mathbb{R}$, *the* Fourier transform *of* $f : \mathbb{R} \to \mathbb{C}$.

(i) *Show that*

$$\hat{f}(\xi) = \frac{1}{(2\pi i\xi)^2}\int_{\mathbb{R}} f''(x)e^{-2\pi i x\xi}dx.$$

(ii) *Show that* $\hat{f} \in L^1(\mathbb{R})$, *that is, show that* $\int_{\mathbb{R}} |\hat{f}(\xi)|d\xi < +\infty$.