Mat-1.3652 Finite Difference Methods, fall 2013

Final exam, 14-17, October 22 **Hyvönen/Majander**

1. Consider the *linear multistep method* (LMM)

$$3x_{j+2} - 4x_{j+1} + x_j = 2hf_{j+2}, \qquad j = 0, 1, 2, \dots . \tag{1}$$

   (a) Is (1) an explicit or an implicit LMM?

   (b) Is (1) consistent (of any order $p > 0$)?

   (c) Is (1) zero-stable?

   Justify your answers. Explain briefly (in one or two sentences) why consistency and zero-stability are indispensable properties of any reasonable LMM.

2. Apply (1) to the model problem

$$x'(t) = \lambda x(t), \qquad x(0) = x_0 \neq 0,$$

   where $\mathbb{R} \ni \lambda < 0$, with a starting value $x_1 = x_1(h)$ satisfying $x_1 \to x_0$ as $h \to 0$. For what values of the time step size $h > 0$ does it hold that

$$\lim_{j\to\infty} x_j = \lim_{t\to\infty} x(t) \ ?$$

   Based on your observations, is it possible that the method (1) is A-stable?

3. Consider the *Runge–Kutta* (RK) method

$$\begin{aligned} x_{j+1} &= x_j + \frac{1}{2}h(k_1 + k_2), \\ k_1 &= f(t_j, x_j), \\ k_2 &= f\Big(t_j + \frac{1}{2}h, x_j + \frac{1}{2}hk_1\Big). \end{aligned} \tag{2}$$

   (a) Is (2) an explicit or an implicit RK method? Why?

   (b) What is the region of absolute stability for (2)?

   (c) Prove that (2) is consistent of order $p = 1$ but not of order $p = 2$.

4. Consider the initial value problem

$$u'(t) = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} u(t), \qquad u(0) = u_0. \tag{3}$$

   (a) Prove that

$$\lim_{t\to\infty} u(t) = 0 \in \mathbb{R}^2$$

   for any $u_0 \in \mathbb{R}^2$.

   (b) For what values of the time step $h > 0$ does the numerical solution produced by the RK method (2) satisfy

$$\lim_{j\to\infty} u_j = 0 \in \mathbb{R}^2$$

   for any $u_0 \in \mathbb{R}^2$. (If you were not able to deduce the answer to 3(b), explain how you would approach this problem setting if you knew the region of absolute stability for (2).)