Quantum Physics TFY-3.4323
Second midterm examination
8th of April 2014

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Please write your answers either in English, Finnish, Swedish, or German.
No calculators allowed.

1. Explain briefly (you may use equations if you find it convenient, but it is not necessary)

   a) Qubit (2 points)

   b) Quantum phase transition (2 points)

   c) Fermi liquid (2 points)

2. Consider non-interacting bosons in a cube of volume $V = L^3$, in which L is the length of the cube. Show that there is Bose-Einstein condensation for certain particle numbers and temperatures.

   Guidelines: The bosons are described by the Hamiltonian

   $$H = \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}) \left(-\frac{\hbar^2\nabla^2}{2m}\right) \psi(\mathbf{x}).$$

   Assuming a periodic boundary condition, the single particle eigenstates are

   $$\varphi_{\mathbf{n}}(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}_{\mathbf{n}}\cdot\mathbf{x}}$$

   with the wave-vector $\mathbf{k}_{\mathbf{n}} = \frac{2\pi}{L}(n_x, n_y, n_z)$, $n_i = 0, \pm 1, \pm 2, \ldots$ At an inverse temperature $\beta = \frac{1}{k_B T}$ and chemical potential μ the expected number of particles in state $\varphi_{\mathbf{n}}$ is

   $$n_{\mathbf{n}} = \langle \hat{a}^\dagger_{\mathbf{n}} \hat{a}_{\mathbf{n}} \rangle = \mathrm{Tr}\{\hat{\rho}\hat{a}^\dagger_{\mathbf{n}}\hat{a}_{\mathbf{n}}\} = \frac{ze^{-\beta E_{\mathbf{n}}}}{1 - ze^{-\beta E_{\mathbf{n}}}},$$

   with the notation $z = e^{\beta\mu}$. The total density of particles in the system is

   $$\frac{N}{V} = \frac{1}{V}\sum_{\mathbf{n}} n_{\mathbf{n}} = \frac{1}{V}\sum_{\mathbf{n}} \frac{ze^{-\beta E_{\mathbf{n}}}}{1 - ze^{-\beta E_{\mathbf{n}}}}.$$

   You may find it useful to estimate a discrete sum with an integral, to employ spherical coordinates and to use the following function

   $$g_{\frac{3}{2}}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx\, x^2 \frac{ze^{-x^2}}{1 - ze^{-x^2}}, \qquad g_{\frac{3}{2}}(1) = 2.612\ldots$$

TURN THE PAPER!