3. Using the Lagrangian density

$$\mathcal{L}[\phi_0, \phi_0^*, \nabla\phi_0, \nabla\phi_0^*] = \frac{\hbar^2}{2m}(\nabla\phi_0^*(\mathbf{r}))\cdot\nabla\phi_0(\mathbf{r}) + (V_T(\mathbf{r}) - \mu)\,|\phi_0(\mathbf{r})|^2 + \frac{2\pi\hbar^2 a_s}{m} N|\phi_0(\mathbf{r})|^4$$

and the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial(\phi_0^*(\mathbf{r}))} - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi_0^*(\mathbf{r}))}\right) = 0$$

derive the Gross-Pitaevskii equation.

4. Starting from the Hamiltonian

$$H = \int d\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(\mathbf{r})\left(-\frac{\hbar^2\nabla^2}{2m} - \mu\right)\psi_\sigma(\mathbf{r}) + V_0\int d\mathbf{r}\,\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})$$

for interacting fermions, do the BCS mean field approximation and explain the meaning of the Pairing, Hartree and Fock fields. Write down the mean field Hamiltonian and express it in momentum basis using

$$\psi_\sigma(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r})c_{\sigma\mathbf{k}}.$$

Here m and μ are the mass and the chemical potential of the particle described by the field ψ_σ, V_0 is the interaction strength in the contact potential approximation and V is the volume of the system.