Problem 3 (8pt)

Consider the Assignment Problem

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, ..., n \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, ..., n \\
& x_{ij} \geq 0, \quad i, j = 1, ..., n.
\end{aligned}$$

Use the primal-dual algorithm to solve the following instance

$$(c_{ij}) = \begin{pmatrix} 7 & 9 & 8 & 9 \\ 2 & 8 & 5 & 7 \\ 1 & 6 & 6 & 9 \\ 3 & 6 & 2 & 2 \end{pmatrix}$$

Start with the dual solution $\mathbf{u} = (0, -1, 0, -3)$, $\mathbf{v} = (7, 9, 8, 9)$. Show the operations at each primal and dual the step. Report the node labels after each primal step, and show how the dual solution is updated in the dual step. Report the optimal assignment together with the optimal dual solution.

Problem 4 (10pt)

Consider the following Uncapacitated Facility Location Problem

$$\begin{aligned}
\text{(IP)} \quad \max \quad & \sum_{i\in M}\sum_{j\in N} c_{ij}x_{ij} - \sum_{j\in N} f_j y_j \\
\text{s.t.} \quad & \sum_{j\in N} x_{ij} \leq 1, \quad \forall i \in M, \\
& x_{ij} - y_j \leq 0, \quad \forall j \in N, \forall i \in M, \\
& x_{ij} \geq 0, \quad \forall j \in N, \forall i \in M, \\
& y_j \in \{0, 1\}, \quad \forall j \in N.
\end{aligned}$$

(a) Dualize the first set of constraints and derive the corresponding Lagrangian subproblem IP($\mathbf{u}$), and the Lagrangian dual. Explain how to solve the Lagrangian subproblem.

(b) Let z_{LP} be the optimal cost of the LP relaxation of IP, and let w_{LD} be the optimal cost of the Lagrangian dual derived in (a). Compare z_{LP} and w_{LD}, and justify your conclusion.