Rak-54.3200 Numerical methods in structural engineering
Examination, April 10, 2014 / Niiranen

This examination consists of 4 problems, each one rated by using the scale 1...6.

Problem 1

Let us consider a long and tall wall with a constant thickness and constant temperature values on both surfaces. A heat source inside the wall can be modelled by a quadratic function

$$f = f(x) = f_0 x(1 - \frac{x}{L}),$$

where x denotes the coordinate along a line across the wall, L denotes the thickness of the wall, and f_0 is a constant.

Accordingly, the temperature distribution inside the wall can be modelled by a one-dimensional stationary heat diffusion problem: For given thermal conductivity $k = k(x)$, heat source $f = f(x)$ and boundary values T_0 and T_L, find the temperature distribution $T = T(x)$ such that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}(k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}) &= f(x) \quad \forall x \in (0, L) \\ T(0) &= T_0 \\ T(L) &= T_L. \end{aligned}$$

(i) Derive the weak form of the boundary value problem.

(ii) Construct a finite element approximation for the problem by applying two linear elements of equal size and assuming a constant conductivity $k = k_0$ as well as zero boundary values $T_0 = 0 = T_L$.

$(uv)' = u'v + v'u$

Problem 2

The bilinear form of the Timoshenko beam problem can be written as

$$a(w, \beta; v, \eta) = \int_\Omega EI\beta'\eta' d\Omega + \int_\Omega GA(w' - \beta)(v' - \eta) d\Omega.$$

$\int u'v = |uv - v'u$

(i) Define and name the quantities, variables, indeces and other notation appearing in the form.

(ii) Derive the corresponding bilinear form of the Euler–Bernoulli beam problem from the bilinear form of the Timoshenko problem above.

(iii) Utilize the bilinear form above and write down the standard conforming finite element formulation for the Timoshenko beam problem – including the bilinear form, load functional and finite element spaces as usual. The beam is assumed to be clamped in both ends and subject to a uniform distributed transversal load. Elements can be assumed to be of the lowest possible order.

Problem 3

For the conforming finite element method of the Euler–Bernoulli beam problem, with certain assumptions the basic mathematical finite element error estimate is of the form

$$\|w - w_h\|_2 \leq Ch^{k-1}|w|_{k+1}.$$

(i) Define and name the quantities, variables, indeces and other notation of the inequality, and shortly describe the information this estimate provides about the approximation properties of the finite element method.

(ii) Let us assume that for a beam problem at hand the finite element approximation follows the error estimate given above. Let us then assume that the first finite element approximation for the problem has been obtained by deviding the solution interval (0,L) into three elements of equal size giving the relative error

$$\frac{\|w - w_h\|_2}{|w|_4} = 0.2.$$

For the next finite element approximation, the mesh will be refined such that each element will be splitted into two elements of equal size. What is your justified estimate for the relative error of the finite element approximation corresponding to this new refined mesh?

(iii) How many elements would be needed for diminishing the relative error below one percent?

$$\|w - w_h\|_m \leq C h^{k+1-m} |w|_{k+1}$$

$3+1-2$

Problem 4

Let us study the problem of elastodynamics of an axially loaded rod. The strong form of the problem is of the following form: For a given distributed loading $b = b(x,t)$ defined in $\Omega \times I$, with intervals $\Omega = (0,L), I = (0,t_1)$, as well as an end point loading $N_L(t)$ and an end point displacement $u_0(t)$ both defined in I, find $u = u(x,t)$ such that

$$\begin{aligned}
\rho(x)A(x)\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}(E(x)A(x)\frac{\partial u(x,t)}{\partial x}) &= b(x,t) \quad \forall (x,t) \in \Omega \times I \\
u(0,t) &= u_0(t) \quad \forall t \in I \\
N(L,t) &= N_L(t) \quad \forall t \in I \\
u(x,0) &= \bar{u}(x) \quad \forall x \in \Omega \\
\frac{\partial u}{\partial t}(x,0) &= \bar{v}(x) \quad \forall x \in \Omega.
\end{aligned}$$

Above, ρ, A and E stand for the material and geometrical properties of the rod as usual, while the axial force is denoted by

$$N(x,t) = E(x)A(x)\frac{\partial u(x,t)}{\partial x}.$$

For the strong form above, the corresponding weak form can be written in the following form: Find $u = u(x,t)$, $u(\cdot,t) \in H^1(\Omega)$, satisfying $u(0,t) = u_0(t)$ and

$$\begin{aligned}
\int_0^L \rho A \ddot{u}\hat{u}\,\mathrm{d}\Omega + \int_0^L EAu'\hat{u}'\,\mathrm{d}\Omega &= \int_0^L b\hat{u}\,\mathrm{d}\Omega + N_L(t)\hat{u}(L) \\
\int_0^L \rho A u(0,\cdot)\hat{u}\,\mathrm{d}\Omega &= \int_0^L \rho A \bar{u}\hat{u}\,\mathrm{d}\Omega \\
\int_0^L \rho A \dot{u}(0,\cdot)\hat{u}\,\mathrm{d}\Omega &= \int_0^L \rho A \bar{v}\hat{u}\,\mathrm{d}\Omega
\end{aligned}$$

for all $\hat{u} = \hat{u}(x)$, $\hat{u} \in H^1(\Omega)$, satisfying $\hat{u}(0) = 0$. Above, the notations

$$\dot{u} = \frac{\partial u}{\partial t}, \quad \ddot{u} = \frac{\partial^2 u}{\partial t^2}$$

have been used for the partial derivatives with respect to time t.

Derive the corresponding equation system of the semidiscrete finite element formulation by starting from the weak form above and introducing the shape function approximations of the displacement u and the corresponding test function $\hat{u}$ – and so on.