4. Let us define the so called quadratures of a quantized electromagnetic field for a single frequency ω

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger).$$

The quantum Hamiltonian of the field is

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a},$$

and the classical Hamiltonian is

$$H_{\mathrm{cl}} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2.$$

The coherent state $|\alpha\rangle$ is defined by demanding that the expectation value of the quantum Hamiltonian in this state equals the classical Hamiltonian, when p and q in the classical Hamiltonian are replaced by the expectation values of $\hat{p}$ and $\hat{q}$ in the state $|\alpha\rangle$. Using this requirement show that

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \qquad \langle\alpha|\,\hat{a}^\dagger = \langle\alpha|\,\alpha^*.$$

With this, express then the expectation value of the quantum Hamiltonian in the coherent state in terms of $\hbar$, ω and $|\alpha|^2$. What is the physical meaning of $|\alpha|^2$? Discuss briefly the phase and amplitude (photon number) uncertainty in the coherent state. Give some well known example of a coherent state.

5. To describe e.g. bosonic atoms in an optical lattice one may use the Bose-Hubbard Hamiltonian

$$H = -J\sum_{<i,j>} a_i^\dagger a_j + \frac{U}{2}\sum_i a_i^\dagger a_i^\dagger a_i a_i$$

together with the Gutzwiller mean-field ansatz

$$|\Psi_{MF}\rangle = \prod_{i=1}^{M}\left[\sum_{n_i=0}^{\infty} f_{n_i}^{(i)}|n_i\rangle\right].$$

a) Calculate the following quantities:

$$\langle\Psi_{MF}|H|\Psi_{MF}\rangle,$$
$$\langle\Psi_{MF}|\hat{n}_i|\Psi_{MF}\rangle,$$
$$\langle\Psi_{MF}|a_i|\Psi_{MF}\rangle,$$
$$\sigma_i^2 = \frac{\langle\Psi_{MF}|\hat{n}_i^2|\Psi_{MF}\rangle - \langle\Psi_{MF}|\hat{n}_i|\Psi_{MF}\rangle^2}{\langle\Psi_{MF}|\hat{n}_i|\Psi_{MF}\rangle}.$$

(We have the notation $\hat{n}_i = a_i^\dagger a_i$.)

b) Explain briefly how you can distinguish between a Mott insulator and a superfluid phase based on the quantities above.