

AS-84.3127 Paikannus- ja navigointimenetelmät

Tentti 05.3.2013

Tentissä on lupa käyttää **T. Pentikäinen, Matematiikan kaavoja tai MAOL matematiikan taulukot** kaavakokoelmaa sekä tentin kanssa jaettuja kaavoja.

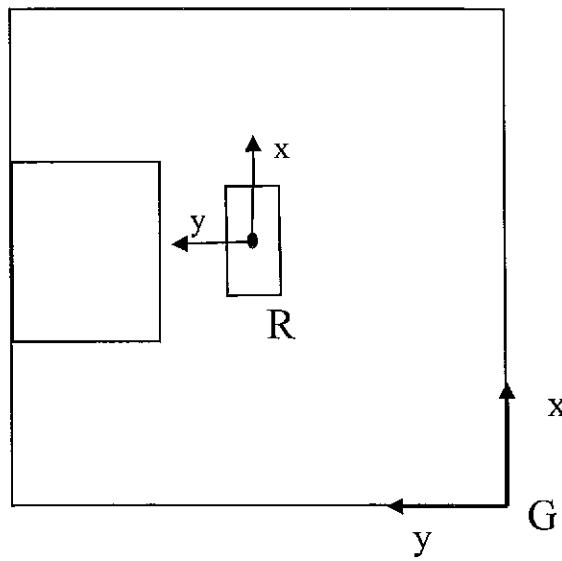
- 1.) Liikkuvien robottien navigointimenetelmät voidaan jakaa kahteen pääluokkaan: reaktiiviset liikeradansuunnittelumenetelmät sekä karttaan pohjautuvat liikeradansuunnittelumenetelmät. Selosta kummankin navigointitavan toimintaperiaate sekä niiden edut että epäkohdat. (6p)

- 2.) Moottoripyörä liikkuu pohjois-itäsuuntaisessa xy-koordinaatistossa. Kulkusuunta ϕ määritetään kompassilla x-akselista myötäpäivään. Etupyörän ohjauskulma on positiivinen käännyttäessä oikealle. Kirjoita moottoripyörän taka-akselin nopeuskomponenttien ja suuntakulman differentiaaliyhälöt pohjois (x)-itäsuunnassa (y). Piirrä myös globaalikoordinaatisto ja moottoripyöräkoordinaatisto sekä määrittele kulmien suunta koordinaatistoissa. Moottoripyörän nopeus v ja etupyörän ohjauskulma u tunnetaan. Moottoripyörä lähee origosta. (6p)

- 3.) Liikkuvan robotin ulkoisilta mitta-antureilta (esim. laser-skanneri ja kamera) saatavaan mittadataan pohjautuvat paikannusmenetelmät voidaan ryhmitellä tunnistettaviin maamerkeihin ja datan sovitukseen pohjautuviin menetelmiin. Selosta kumpaankin luokkaan kuuluvien menetelmien toimintaperiaate, keskeiset vaiheet ja osatehtävä.
 - a) Maamerkeihin pohjautuvat paikannusmenetelmät. (3p)
 - b) Datan sovitukseen pohjautuvat paikannusmenetelmät. (3p)

- 4.) Alla olevassa kuvassa on esitetty robotti, jonka toimintaympäristönä on huone. Robotti määritää aluksi etäisydet oman koordinaatistonsa suhteeseen sekä seinien suuntakulmat oman koordinaatistonsa x-akselista vastapäivään. Tavoitteena on että robotti pystyisi määrittämään paikkansa ja suuntansa myös kulmaan kiinnitetyyn referenssikoordinaatioston G suhteeseen.
 - a) Selosta sanallisesti miten robottiin kiinnitetyn, tasossa mittaavan laserskanterin peräkkäisistä etäisyysmittauksista (361 kpl) lasketaan kulmahistogrammi vastaten kuvassa esitettyä robotin paikkaa ja asentoa. Missä kohtaa histogrammia ovat kunkin seinän/kaapin paikkaa vastaavat huiput? Robotin skannaussektori on 180 astetta robotin y-akselin suunnasta alkaen myötäpäivään. (3p)

 - b) Selosta sanallisesti miten x- ja y-koordinaattien suuntaiset etäisyys histogrammit määritetään sen jälkeen kun laserin osumapisteiden x- ja y-koordinaateista on "kompensoitu" a)-kohdassa määritetty kiertokulman vaikutus pois. Huoneen koko on noin 3mx3m. Missä kohtaa histogrammia on kunkin seinän/kaapin paikka? (3p)



5.) Junan veturi liikkuu suoria kiskoja pitkin, joten vapausasteita on yksi. Veturin tilasuureina ovat paikka ja nopeus. Kirjoita kaksi differentiaaliyhtälöä Kalman suodinta varten, jotka kuvaavat veturin tiloja. Junan paikkaa ja nopeutta voidaan mitata GPS vastaanottimella 1 Hz taajuudella. Kirjoita myös diskreetti matriisimuotoinen mittausyhtälö Kalman suodinta varten. Veturin kiihtyvyyden keskihajonta on $0,5 \text{ m/s}^2$. GPS paikkamittauksen virheen keskihajonta 5 m ja nopeusmittauksen keskihajonta on 0,1 m/s. Kirjoita diskreetin mittausmallin ja jatkuva-aikaisen tilamallin virheiden diagonaaliset kovarianssimatriisit Q ja R. (6p)

Kaavoja ilman indeksejä:

$$dx/dt = A*x + B*u + w \quad Q = E\{w*w^T\}$$

$$y = C*x + v \quad R = E\{v*v^T\}$$

Kalman suodin (ilman indeksejä)

$$dx/dt = A*x + B*u$$

$$dP/dt = A*P + P*A^T + Q$$

$$x = x + K*(y - C*x)$$

$$P = P - P*C^T(C*P*C^T + R)^{-1}C*P$$

$$K = P*C^T(C*P*C^T + R)^{-1}$$

AS-84.3127 Positioning and navigation methods

Examination 5.3.2013

It is allowed to use following mathematical tables in the examination **T. Pentikäinen, Matematiikan kaavoja** or **MAOL matematiikan taulukot** or the collection of equations delivered with the exam.

1.) Navigation methods of mobile robots can be divided in two main categories: reactive navigation and map-based navigation (i.e. path planning) methods. Explain the operational principle of the two main approaches as well as their benefits and drawbacks. (6p)

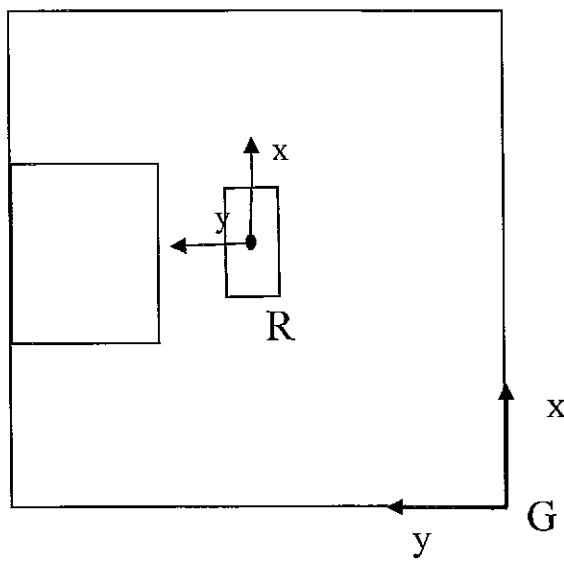
2.) A motorbike moves in North-East xy-coordinates. Heading ϕ clockwise from x-axis is obtained from a magnetic compass. The front wheel angle is positive at right turn. Write the differential equations of the heading and the rear axis velocity components at North (x)-East (y) coordinates. Sketch the coordinates and define angles in the coordinates. Motorbike speed v and front wheel angle u are known. Motorbike starts from the origo. (6p)

3.) Mobile robot localization methods, which are based on external perception sensor (e.g. laser range finder or camera) data can be divided in two main categories: landmark-based localization methods and methods based on matching a local data set, acquired from the perception sensors, with a global map. Explain the general procedure and subtasks of the methods in both categories.

- a) Landmark-based localization methods. (3p)
- b) Map-matching/map-based positioning methods. (3p)

4.) A robot is shown in the following figure. The working environment is a room. In the beginning the robot measures the wall distances and the wall angles relative to the robot's own coordinates. The aim is for the robot to compute its heading and position on the coordinates G fixed on the room corner. The robot computes heading anticlockwise from its x-axis.

- a) Explain how the angle histogram is computed from the sequential laser range measurements. Where in the histogram are the peaks corresponding to each particular wall/box? The laser scanner is fixed on the robot and the scanning sector is 180 degrees clockwise from the y-axis of the robot. (3p)
- b) By using the result of a) the robot is rotated so that the direction of robot x-axis and the direction of the x-axis of the reference coordinates are equal. The room is about 3m x 3m. Explain how the distance histograms in the x and y direction are computed. Where in the histogram are the peaks corresponding each particular wall/box? (3p)



5.) Locomotive is moving on the trails in one degree of freedom. The states of the locomotive are position and velocity. Write two differential equations for the states of the locomotive. The position and velocity of the locomotive is obtained from a GPS receiver at 1 Hz. Write also a discrete matrix form measurement equation for Kalman filter. The standard deviation of the locomotive acceleration is equal to $0,5 \text{ m/s}^2$. The standard deviation of GPS position and velocity measurements are equal to 5 m and $0,1 \text{ m/s}$, respectively. Write the diagonal covariance matrix for the discrete measurement model error and for the continuous system model error. (6p)

Equations without indexes:

$$dx/dt = A*x + B*u + w \quad Q = E\{w*w^T\}$$

$$y = C*x + v \quad R = E\{v*v^T\}$$

Kalman filter (without indexes)

$$dx/dt = A*x + B*u$$

$$dP/dt = A*P + P*A^T + Q$$

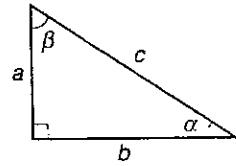
$$x = x + K*(y - C*x)$$

$$P = P - P*C^T * (C*P*C^T + R)^{-1} * C*P$$

$$K = P*C^T * (C*P*C^T + R)^{-1}$$

Triangle trigonometry

Trigonometric formulas, see sec. 5.4.



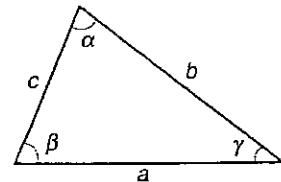
1. Right triangle.

$$(1) \quad \sin \alpha = \frac{a}{c}, \quad \cos \alpha = \frac{b}{c}, \quad \tan \alpha = \frac{a}{b}, \quad \cot \alpha = \frac{b}{a}$$

2. General triangle.

$$(2) \quad \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R} \quad (\text{law of sines})$$

$$(3) \quad a^2 = b^2 + c^2 - 2bc \cos \alpha \quad (\text{law of cosines})$$



$$\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}} \quad (\text{law of tangents})$$

$$A = \frac{bc \sin \alpha}{2} \quad (\text{area formula})$$

$$(4) \quad \alpha + \beta + \gamma = 180^\circ \quad (\text{angle relation})$$

Solution of plane triangles

1. Right triangle: Use (1).

2. General triangle:

	Given		Method: Find
1.	Three sides	a, b, c	α, β, γ from (3) and (4)
2.	Two sides and the included angle	b, c, α	a from (3); β (if $b < c$) from (2); γ from (4)
3.	Two sides and an opposite angle	b, c, β	γ from (2); α from (4); a from (2). (Possibly two solutions)
4.	One side and two angles	a, β, γ	α from (4); b, c from (2)

8.5 Taylor Series

Taylor's formula

Let $f(x)$ and its $n+1$ first derivatives be continuous in an interval about $x=a$. Then in this interval:

$$(8.2) \quad f(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + R_{n+1}(x),$$

where $R_{n+1}(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$,
 $(\xi \text{ between } a \text{ and } x)$

Maclaurin's formula

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots + \frac{f^{(n)}(0)}{n!} x^n + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x), \quad (0 < \theta < 1)$$

Note: $f(x)$ is $\begin{cases} \text{odd: only odd powers of } x \\ \text{even: only even powers of } x. \end{cases}$

Taylor's series

If $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$ then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \quad [\text{Taylor series}]$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad [\text{Maclaurin series}]$$

The Ordo concept (Big O and Little o)

- $f(x) = O(x^a)$ as $x \rightarrow 0$ means: $f(x) = x^a H(x)$, where $H(x)$ is bounded in a neighbourhood of $x=0$.
- $f(x) = o(x^a)$ as $x \rightarrow 0$ means: $f(x)/x^a \rightarrow 0$ as $x \rightarrow 0$.

$$1. \quad O(x^4) \pm O(x^4) = O(x^4)$$

$$2. \quad O(x^3) \pm O(x^4) = O(x^3)$$

$$3. \quad x^2 O(x^3) = O(x^5) = x^5 O(1)$$

$$4. \quad O(x^2) O(x^3) = O(x^5)$$

Corresponding rules for little o .

Example.

$$e^x = 1 + x + \frac{x^2}{2} + \left\{ \begin{array}{l} O(x^3) \\ o(x^2) \end{array} \right\} = 1 + x + \frac{x^2}{2} + \left\{ \begin{array}{l} x^3 O(1) \\ x^2 o(1) \end{array} \right\}$$

6.3

Integrals

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad \frac{d}{dx} \int_x^a f(t) dt = -f(x)$$

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) dt = f(v(x))v'(x) - f(u(x))u'(x)$$

$$\frac{d}{dx} \int_{u(x)}^{v(x)} F(x, t) dt = F(x, v) \frac{dv}{dx} - F(x, u) \frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} F(x, t) dt$$

Inverse function

$$\frac{dx}{dy} = \left(\frac{dy}{dx} \right)^{-1} \quad \frac{d^2 x}{dy^2} = - \frac{d^2 y}{dx^2} \left(\frac{dy}{dx} \right)^3 \quad \frac{d^3 x}{dy^3} = \left[3 \left(\frac{d^2 y}{dx^2} \right)^2 - \frac{dy}{dx} \frac{d^3 y}{dx^3} \right] \left(\frac{dy}{dx} \right)^5$$

Implicit function

$y = y(x)$ given implicitly by $F(x, y) = 0$:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \quad \frac{d^2 y}{dx^2} = -\frac{1}{F_y^3} [F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2]$$

Basic derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^a	ax^{a-1}	$\sinh x$	$\cosh x$	$\sin x$	$\cos x$
$\frac{1}{x^a}$	$-\frac{a}{x^{a+1}}$	$\cosh x$	$\sinh x$	$\cos x$	$-\sin x$
\sqrt{x}	$\frac{1}{2\sqrt{x}}$	$\tanh x$	$\frac{1}{\cosh^2 x} = 1 - \tanh^2 x$	$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\coth x$	$-\frac{1}{\sinh^2 x} = 1 - \coth^2 x$	$\cot x$	$-\frac{1}{\sin^2 x} = -1 - \cot^2 x$
$\frac{1}{x^2}$	$-\frac{2}{x^3}$	$\text{arsinh } x$	$\frac{1}{\sqrt{x^2+1}}$	$\sec x$	$\sin x \sec^2 x$
e^x	e^x	$\text{arcosh } x$	$\frac{1}{\sqrt{x^2-1}}$	$\csc x$	$-\cos x \csc^2 x$
a^x	$a^x \ln a$	$\text{artanh } x$	$\frac{1}{1-x^2}$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\ln x $	$\frac{1}{x}$	$\text{arcoth } x$	$\frac{1}{1-x^2}$	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$a \log x $	$\frac{1}{x \ln a}$			$\arctan x$	$\frac{1}{1+x^2}$
				$\text{arccot } x$	$-\frac{1}{1+x^2}$

6.3

4. *piece-wise continuous* in $[a, b]$, if $f(x)$ is continuous at all but a finite number of points and that, at each point of discontinuity, the one-sided limits of $f(x)$ exist and are finite.

Theorems

1. f, g continuous $\Rightarrow f \pm g, fg, f/g, f \cdot g$ continuous (where they are defined).
2. Any composition of the elementary functions is continuous where it is defined.
3. $f(x)$ continuous on a closed interval $[a, b] \Rightarrow$
 - (a) $f(x)$ assumes every value between $f(a)$ and $f(b)$.
 - (b) $f(x)$ assumes its supremum (greatest value) and its infimum (least value) in $[a, b]$.
 - (c) $f(x)$ is bounded in $[a, b]$
 - (d) $f(x)$ is uniformly continuous in $[a, b]$.
4. $f'(x)$ bounded in an interval $I \Rightarrow f(x)$ uniformly continuous in I .
5. $f'(x)$ exists on a closed interval $[a, b] \Rightarrow$
 $f'(x)$ assumes every value between $f'(a)$ and $f'(b)$.

6.3 Derivatives

The *derivative* $f'(x)$ of a function $y=f(x)$ is defined by

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

Alternative notation:

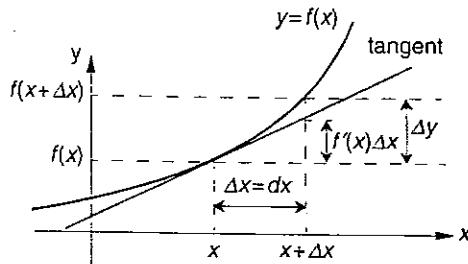
$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x),$$

$$\dot{y} = \frac{dy}{dt} \quad (\text{y a function of time})$$

Higher derivatives

$$y'' = f''(x) = \frac{d^2 y}{dx^2} = D^2 f(x),$$

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = D^n f(x) = [\text{def: } D\{D^{n-1} f(x)\}].$$



Differential

Difference $\Delta f = f(x + \Delta x) - f(x)$

Differential $df = f'(x)dx$

$$f(x) \text{ differentiable} \Rightarrow \Delta f = f'(x)\Delta x + \varepsilon(x)\Delta x, \text{ where } \varepsilon(x) \rightarrow 0 \text{ as } \Delta x \rightarrow 0$$

Equation of *tangent* of curve $y = f(x)$ at $(a, f(a))$: $y - f(a) = f'(a)(x - a)$

Equation of *normal* of curve $y = f(x)$ at $(a, f(a))$: $y - f(a) = -\frac{1}{f'(a)}(x - a)$

Matrix algebra

1. Addition $C = A + B : c_{ij} = a_{ij} + b_{ij}$ (A, B, C of the same order)
2. Subtraction $C = A - B : c_{ij} = a_{ij} - b_{ij}$ (A, B, C of the same order)
3. Multiplication by a number, $C = xA : c_{ij} = xa_{ij}$ (A, C of the same order)
4. Product AB of two matrices:

If order $(A) = m \times n$, order $(B) = n \times p$, then $C = AB$ is of order $m \times p$

and

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} \quad i \begin{bmatrix} & & & & & A \\ & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} \begin{bmatrix} & & & & & B \\ & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & j \end{bmatrix} = i \begin{bmatrix} & & & & & C \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & j \end{bmatrix}$$

Note: $AB = A[\mathbf{b}_1, \dots, \mathbf{b}_p] = [A\mathbf{b}_1, \dots, A\mathbf{b}_p]$

$$A + B = B + A \quad (A + B) + C = A + (B + C) \quad x(A + B) = xA + xB$$

$$AB \neq BA \text{ (in general)} \quad (AB)C = A(BC) \quad IA = AI = A$$

$$A(B + C) = AB + AC \quad (A + B)C = AC + BC \quad (AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad (A^{-1})^T = (A^T)^{-1} \quad (A + B)^T = A^T + B^T$$

$$e^{A+B} = e^A e^B \text{ if } AB = BA \quad (e^A)^{-1} = e^{-A} \quad \frac{d}{dx} e^{xA} = A e^{xA}$$

Differentiation

If $A = A(x) = (a_{ij}(x))$ and $B = B(x)$, then (i) $A'(x) = (a'_{ij}(x))$ (ii) $(A + B)' = A' + B'$ (iii) $(AB)' = AB' + A'B$ (iv) $(A^2)' = AA' + A'A$ (v) $(A^{-1})' = -A^{-1}A'A^{-1}$

Matrix norms, see sec. 16.2.

Rank

Elementary row operations

- I. Exchange of two rows.
- II. Multiplication of a row by a constant $\neq 0$.
- III. Addition of an arbitrary multiple of a row to another row.

Notation: $A \sim B$ if A can be transformed to B by a sequence of elementary row operations.