Aalto University
Department of Information and Computer Science
Shinnosuke Seki

T-79.5205 Combinatorics (5 cr)
Midterm exam, Monday 20th October 2014, 16–19

Write down on each answer sheet:
- Your name, degree programme, and student number
- The text: "T-79.5205 Combinatorics 20.10.2014"
- The total number of answer sheets you are submitting for grading

Note: You can write down your answers in either Finnish, Swedish, or English.

1. Principle of inclusion and exclusion (Total 6p)

   (a) Give a careful description of the principle of inclusion and exclusion. (2p)

   (b) Using this principle, give the number of prime numbers that are smaller than 100 (note that 1 is not prime). (2p)

   (c) Using this principle, give the number of surjective functions from the n-set $[n] = \{1, 2, \ldots, n\}$ to the k-set $[k] = \{1, 2, \ldots, k\}$. (2p)

2. "*Genjikō*" (Total 6p)
   Genjikō is a game in Japanese *Kodō* (way of fragrance) in which participants are to determine which of five prepared censers (vessels made for burning incense) contain different scents, and which contain the same scent (from wikipedia article of "*Kōdō*"). If you think all of them are pairwise distinct, for instance, your answer is written as $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$.

   (a) How should we describe the answer according to this format when we think the first three are the same, and the other two are also the same but distinct from the first three? (1p)

   (b) Let's generalize Genjikō to n censers, and by $B(n)$, we denote the number of possible answers in this generalized Genjikō. We set $B(0) = 1$. Give a recurrence relation for $B(n)$ in terms of $B(0), B(1), \ldots, B(n-1)$. (2p)

   (c) Using the relation just obtained, find $B(5)$, i.e., the number of possible answers in the original (5-censers) Genjikō. (1p)

   (d) Give a closed-form expression for $B(n)$. (2p)
   Hint: describe the previously-obtained number of surjective mappings from $[n]$ to $[k]$ in terms of Stirling partition numbers $S(n, k)$ and consider the definition of $B(n)$.

3. (Partially) ordered set (Total 12p)

   (a) Prove that every nonempty finite ordered set $(S, \leq)$ has a unique maximal element $m \in S$ (that is, no element $a \in S$ satisfies $m < a$). (2p)

   (b) A permuation π over an n-set $[n] = \{1, 2, \ldots, n\}$ is represented as an n-dimensional vector $(\pi(1), \pi(2), \ldots, \pi(n))$. Give the number of permutations over $[n]$. (2p)

(c) In the case $n = 4$, list all such permutations in the lexicographic order. (2p)

(d) Consider the lexicographically-ordered list of the permutations of $[n]$. If a permutation π of $[n]$ is the t-th element in this list, then the *rank* of π is t. The rank of the minimum element is defined to be 1. Give the closed formula for the rank of π. (3p)

(e) Explain how to unrank the k-th element of this list, that is, how to compute the k-th element. (3p)