Aalto ME-C3100 Computer Graphics, Fall 2014

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Välikoe/midterm, 24.10.2014

Allowed: One two-sided A4 sheet of notes, calculators (also symbolic). Turn your notes in with your answers. Write your answers in either Finnish or English.

Name, student ID: _____

1 Linear Algebra and Transformations [/ 11]

1.1 Linearity [/ 2]

What properties characterize a linear function (operator) $L(x)$, with $x \in \mathbb{R}^n$? Write them down in one or two equations.

1.2 Linear vs. Rigid [/ 2]

How does a linear transformation differ from a rigid (Euclidean) transformation? Specifically, it is enough to name one transformation that is linear but not rigid, and one transform that is rigid but not linear.

1.3 Groups [/ 2]

The types of transformations we've looked at form *groups (ryhmä)*. What does this mean and why is it useful? You do not need to give a formal definition.

1.4 Affine Matrices [/ 4]

An $n \times n$ matrix $\mathbf{M}$ is called an *affine matrix* if it is of the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}, \tag{1}$$

where $\mathbf{A}$ is an $(n-1) \times (n-1)$ matrix, $\mathbf{b}$ is an $(n-1) \times 1$ column vector, $\mathbf{0}$ is a $1 \times (n-1)$ row vector of zeros (and 1 is a scalar). Show that the product $\mathbf{M}_1 \mathbf{M}_2$ of two affine matrices is still an affine matrix by writing the product out explicitly, including the intermediate step. *Note that you can do this without going into individual scalar components. Rather, the solution is simplest if you write the block structure of the result in terms of the $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{0}$ and 1. Writing out the components is much more complicated!*

1.5 Homogeneous Coordinates [/ 1]

Consider the homogeneous 2D point (x, y, w). What is the geometric interpretation of the scaling (sx, sy, sw), with $s > 0 \in \mathbb{R}$ when we think about what actual 2D point this homogeneous point represents? Remember projective equivalence $(wx, wy, w) \equiv (x, y, 1)$ when $w \neq 0$.

2 Hierarchical Modeling [/ 4]

2.1 Forward and Inverse Kinematics

Briefly describe what is meant by forward and inverse kinematics (suora ja käänteinen kinematiikka). What is the role of joint angles (nivelkulma) in each case? [/ 4]

3 Curves, Splines [/ 8]

3.1 Tangent and Curvature [/ 2]

The curvature vector $\mathbf{K}$ for a curve $\mathbf{P}(t)$ is defined as the derivative of the unit tangent $\mathbf{T} = \mathbf{P}'(t)/\|\mathbf{P}'(t)\|$, i.e., $\mathbf{K} = \mathbf{T}'$, where the prime denotes differentiation w.r.t. t. Show that $\mathbf{K}$ is always orthogonal to $\mathbf{T}$. You don't need to consider pathological cases such as when $\mathbf{P}' = 0$[1]. **Hint:** $\mathbf{T} \cdot \mathbf{T} \equiv 1$. Apply the elementary product rule for differentiation.

3.2 Derivatives of Bézier Curves [/ 6]

A cubic Bézier curve is given by

$$\mathbf{P}(t) = \mathbf{G} \mathbf{B} \mathbf{T}(t) = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{P}_3 & \mathbf{P}_4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix},$$

where the $\mathbf{P}_n \in \mathbb{R}^n$ are the control points. Write down a similar matrix equation for the derivative $\mathbf{P}'(t)$. **Hint:** Only the spline matrix should change. What is the new matrix in place of $\mathbf{B}$? There are at least two ways of doing this: one is expanding out the Bernstein polynomials $\mathbf{B}\mathbf{T}(t)$, differentiating them, collecting the terms, and regrouping them into a new matrix $\mathbf{B}'$. But there is an even simpler way.

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[1]or even recall what tangents and curvatures mean other than what's said here. The problem statement contains all you need to know in addition to elementary calculus and vector analysis.

4 Physically-Based Animation [/ 7]

4.1 The Euler Method and Springs [/ 7]

Consider a unit mass (red circle) attached to the origin with a massless *damped spring* that is constrained to always lie along the x axis, and denote the position of the mass at time t with $x(t)$:
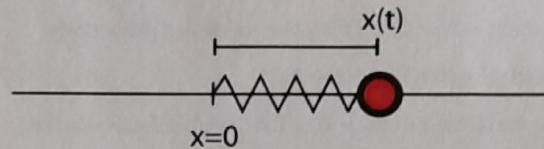
The motion of the spring is described by the differential equation

$$\frac{d^2 x}{dt^2} = -k(x - 2) - l\frac{dx}{dt}$$

where the constants k, l are positive.

a) This is a second-order ordinary differential equation (ODE). Write down the corresponding first-order system of differential equations with variables x and $v = dx/dt$. *Yes, this is easy. If your result is complicated, you've gone astray!* [/ 2]

b) What is the rest length of the spring? What is the corresponding "rest position" of the free (non-attached) end when the spring is at rest? [/ 2]

c) Let the initial conditions be $x(0) = 2$ and $\frac{dx}{dt}(0) = 3$. What is the state of the system after one timestep of Euler integration with step size h? Give the values of x and v. [/ 2]

d) What is the physical significance/interpretation of the term $-l(dx/dt)$? [/ 1]

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5 Extra Credit [/ 16]

Partial credit for the extra credit questions is only given if you make a technical mistake in solving an equation that is otherwise correctly derived. Continue your answers onto the next blank page if necessary.

5.1 Custom cubic spline [/ 10]

Derive the spline matrix for a cubic spline $\mathbf{P}(t)$ with the following properties:

1. it interpolates the first control point $\mathbf{P}_1$ at $t = 0$;

2. its tangent $\mathbf{P}'(t)$ at $t = 0$ matches $3 * (\mathbf{P}_2 - \mathbf{P}_1)$ like a cubic Bézier curve,

3. it interpolates the third and fourth control points such that $\mathbf{P}(2/3) = \mathbf{P}_3$ and $\mathbf{P}(1) = \mathbf{P}_4$.

You will need to write out these constraints as a system of equations — it will turn out to be a linear system — and solve it to get the spline coefficients.

5.2 Implicit Integration [/ 6]

Derive the implicit Euler update rule for the differential equation from Question 4.1. In other words, give the formulae for x_{i+1} and v_{i+1} in terms of x_i and v_i remembering that abstractly, the implicit Euler scheme reads $\mathbf{X}_{i+1} = \mathbf{X}_i + h\,f(\mathbf{X}_{i+1})$ where $f(\mathbf{X})$ is the function that determines the derivative $d\mathbf{X}/dt$ in that state. *In this case, it turns out you can derive explicit formulae that are not very complex. There is no need for Newton iteration or anything like that. After writing things out, you will need to know how to analytically invert a 2×2 matrix, that's pretty much it.*