Aalto University
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T-79.5205 Combinatorics (5 cr)
Final exam, Thursday 11th December 2014, 9–12

Write down on each answer sheet:
- Your name, degree programme, and student number
- The text: "T-79.5205 Combinatorics 11.12.2014"
- The total number of answer sheets you are submitting for grading

Note: You can write down your answers in either Finnish, Swedish, or English.

1. Ramsey Theory (Total 7p)
By $R_r(\underbrace{3,3,\ldots,3}_{r})$, we denote the minimum number n such that, for any complete graph K_n of n vertices and for any r-coloring on its edges, there is a color i such that K_n contains a triangle of color i.
   (a) Prove $R_2(3,3) = 6$. (2p)
   (b) Prove $R_r(3,3,\ldots,3) \le r(R_{r-1}(3,3,\ldots,3)-1)+2$ for any r. (2p)
   (c) By combining (a) and (b), give an upperbound on $R_3(3,3,3)$. (1p)
   (d) **Using probabilistic methods**, prove the lowerbound $R_3(3,3,3) \ge 5$. (2p)

2. Hall's condition (Total 8p)
Let $j_1, j_2, \ldots, j_n$ be jobs, and X be a finite set of processors for these jobs. For $k \in [n] = \{1,2,\ldots,n\}$, by A_k we denote the set of processors in X that can handle the job j_k. A system of distinct representatives (SDR) is an assignment of distinct processors in X to the n jobs.
   (a) Prove that the following Hall's condition is a necessary condition for the existence of SDR. (1p)
$$\forall J \subseteq [n], \left|\bigcup_{k\in J} A_k\right| \ge |J|. \qquad (1)$$
   (b) We will prove the sufficiency of (1) for the existence of SDR using Dilworth's theorem, which says: A finite poset whose largest antichain is of size r can be partitioned into r chains. Assume (1) holds.
      i. Define a poset $(X \cup [n], \le)$ with $x < k$ iff $x \in A_k$ (a processor x is "smaller than" the job j_k iff x can handle j_k). Show that the size of any antichain S of this poset is at most $|X|$. (1p)
      ii. Using the Dilworth's theorem, construct an SDR. (2p)
   (c) Show that, for $d \ge 1$, every bipartite d-regular graph has a perfect matching. (2p)

(d) Take a standard deck of 52 cards. Prove that no matter how you deal them into 13 piles of 4 cards each, you can select exactly 1 card from each pile such that the 13 selected cards contain exactly one card of each rank (ace, 2, 3, ..., queen, king). (2p)

3. Group and symmetry (Total 9p)
Let $(G,*)$ be a group. For a nonempty subset $H \subseteq G$, the pair $(H,*)$ is a *subgroup* of G, written as $H \leq G$, if

- $h_1 * h_2 \in H$ for all $h_1, h_2 \in H$, and
- $h^{-1} \in H$ for all $h \in H$.

(a) Give an example of subgroup and prove that it is a subgroup. (1p)

(b) Let G be a finite group. For each $H, K \leq G$ and $g \in G$, define the double coset as

$$HgK = \{hgk \mid h \in H, k \in K\}.$$

Show that the double cosets $\{HgK \mid g \in G\}$ partition G. (2p)

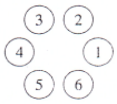
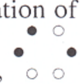
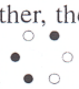
(c) Let D be a finite set and let G be a finite group of functions from D to D (i.e., G acts on D). For $x \in D$,

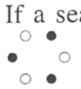
- the *orbit* of x on D is the set $Gx = \{gx \mid g \in G\}$.
- the *stabilizer* of x in G is the set $G_x = \{g \in G \mid gx = x\}$.

For $x \in D$, $|G| = |G_x||Gx|$ (the orbit-stabilizer theorem).
By $G \backslash D$, we denote the set of orbits of G on D. The orbit counting lemma says

$$|G \backslash D| = \frac{1}{|G|} \sum_{g \in G} |F(g)|,$$

where $F(g) = \{x \in D \mid gx = x\}$.
Using the orbit-stabilizer theorem, prove the orbit counting lemma. (2p)

(d) There are 6 chairs around a table like  . Let us count ways for people to take a seat. Note that if one seating is just a rotation of another, then they are not distinguished from each other. For example,  and  are the same.

i. Show that the rotations, together with composition, form a group. (1p)
ii. If a seating is not changed by a rotation, then it is *fixed*; for instance,  is fixed by the rotations by 0, 120, or 240 degree. For each of these rotations, determine the number of seatings which are fixed by the rotation. (1p)
iii. Achieve our goal, that is, count the seatings. (2p)