MS-A0003 Matrix algebra (Aalto University)
Turunen / Saari

Second mid-term exam (10.12.2013, 5pm–8pm)

Please fill in the required information onto each answer sheet.

Calculators and mathematical tables are not allowed.

About grading: Every exam problem will be graded from 0 to 6 points. Harmless small errors do not prevent from getting maximal points. You will get points if your answer contains at least some information (relevant definitions, pictures, calculations etc) — empty answer is surely worth zero.

On notation. In different sources, the Hermitian conjugate (or conjugate transpose) of a matrix $A \in \mathbb{C}^{m\times n}$ is denoted in various ways: for example

$$A^* = A^{\mathrm{H}} = \overline{A^{\mathrm{T}}} \in \mathbb{C}^{n\times m}.$$

Matrix $U \in \mathbb{C}^{m\times m}$ is unitary, if $U^* = U^{-1}$.

1. Let $P = \begin{bmatrix} 1 & 1 \\ s & t \end{bmatrix}$, where real numbers s, t are not equal.

   a) Find $P \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} P^{-1}$.

   b) Diagonalize matrix $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$.

2. Find unitary matrix $U \in \mathbb{C}^{2\times 2}$ such that $D := U^*AU \in \mathbb{C}^{2\times 2}$ is diagonal, where

$$A := \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

   Check that $A = UDU^*$.

3. Find singular value decomposition (SVD) for matrix $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$.
   In other words, find matrices $U, \Sigma, V \in \mathbb{R}^{2\times 2}$ for which $A = U\Sigma V^*$, where U, V are orthogonal (unitary) and
   Σ is the diagonal matrix of singular values.