

1./a)

$$z = x + iy$$

$$\operatorname{Re} \frac{1}{z} = \operatorname{Re} \frac{1}{x+iy} = \operatorname{Re} \frac{x-i'y}{x^2+y^2} = \frac{x}{x^2+y^2}$$

b)

$$z = Re^{i\varphi}$$

$$z^2 + i = 0 \Leftrightarrow (Re^{i\varphi})^2 = -i = e^{i\frac{3\pi}{2}} \Leftrightarrow$$

$$R^2 e^{i2\varphi} = e^{i\frac{3\pi}{2}} \Leftrightarrow$$

$$\begin{cases} R^2 = 1 \\ 2\varphi = \frac{3\pi}{2} + N \cdot 2\pi, N \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\begin{cases} R = 1 \\ \varphi = \frac{3\pi}{14} + N \cdot \frac{2\pi}{7}, N = 0, 1, 2, \dots, 6 \end{cases} \Leftrightarrow$$

$$z = e^{i(\frac{3\pi}{14} + N \cdot \frac{2\pi}{7})} = e^{i(3+4N)\frac{\pi}{14}}, N = 0, 1, 2, \dots, 6$$

c)

$$f(z) = f(x+iy) = e^{-x} (\cos y + i \sin y) = u(x, y) + i v(x, y)$$

$$u(x, y) = e^{-x} \cos y$$

$$v(x, y) = e^{-x} \sin y$$

Olkoon f on analyyttinen pisteessä $z = x + iy$. \Rightarrow

$$\begin{cases} u_x(x, y) = v_y(x, y) \\ u_y(x, y) = -v_x(x, y) \end{cases} \Rightarrow \begin{cases} -e^{-x} \cos y = e^{-x} \sin y \\ -e^{-x} \sin y = e^{-x} \cos y \end{cases} \Rightarrow$$

$$\begin{cases} e^{-x} \cos y = 0 \\ e^{-x} \sin y = 0 \end{cases} \Rightarrow \begin{cases} \cos y = 0 \\ \sin y = 0 \end{cases} \Rightarrow \text{ei ratkaisua.}$$

$\therefore f$ ei ole analyyttinen määritelmäalueella $D \subset \mathbb{C}$.

Esim. $f(z) = e^{-\bar{z}}$

2.) a)

$$u(x, y) = ax^2 + bxy + cy^2, \quad a, b, c \in \mathbb{R}$$

u on harmoninen. (\Leftrightarrow)

$$0 = u_{xx} + u_{yy} = 2a + 2c \Leftrightarrow c = -a$$

$$\therefore u(x, y) = ax^2 + bxy - ay^2, \quad a, b \in \mathbb{R}$$

b)

$$u(x, y) = -3x(y+1)^2 + x^3$$

$$u_{xx} + u_{yy} = 6x - 6x = 0 \Rightarrow u \text{ on harmoninen (C:stä).}$$

$$\begin{cases} v_x = -u_y = 6x(y+1) \\ v_y = u_x = -3(y+1)^2 + 3x^2 \end{cases} \Leftrightarrow$$

$$\begin{cases} v(x, y) = 3x^2(y+1) + C_1(y) \\ v(x, y) = -(y+1)^3 + 3x^2y + C_2(x) \end{cases} \Leftrightarrow$$

$$v(x, y) = 3x^2(y+1) - (y+1)^3 + C, \quad C \in \mathbb{R}$$

Funam.

$$f(z) = f(x+iy) = u(x, y) + i v(x, y)$$

$$= -3x(y+1)^2 + x^3 + i(3x^2(y+1) - (y+1)^3 + C)$$

$$= x^3 + 3x^2i(y+1) + 3xi^2(y+1)^2 + i^3(y+1)^3 + iC$$

$$= (x + i(y+1))^3 + iC = (z + i)^3 + iC, \quad C \in \mathbb{R}$$

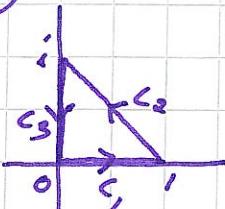
3.) a)

$$C: z(t) = e^{it}, t \in [0, 2\pi]$$

Cauchyn integraalilause
jäi -käsin.

$$\oint_C (z + z^{-1}) dz = \oint_C z dz + \oint_C \frac{1}{z} dz \stackrel{\downarrow}{=} 0 + 2\pi i \cdot 1 = 2\pi i$$

b)



$$C_1: z(t) = t$$

$$\left. \begin{array}{l} C_2: z(t) = 1-t+it = 1+(-1+i)t \\ C_3: z(t) = (1-t)i = i - it \end{array} \right\} t \in [0, 1]$$

$$\oint_C \operatorname{Im} z^2 dz = \oint_C \operatorname{Im} (x+iy)^2 dz = \oint_C \operatorname{Im} (x^2 - y^2 + 2ixy) dz$$

$$= \oint_C 2xy dz = \oint_{C_1} 2xy dz + \oint_{C_2} 2xy dz + \oint_{C_3} 2xy dz$$

$$= \int_0^1 2t \cdot 0 dt + \int_0^1 2(1-t)t(-1+i) dt + \int_0^1 2 \cdot 0 (1-t)(-i) dt$$

$$= 0 + (-2+2i) \int_0^1 (t-t^2) dt + 0$$

$$= (-2+2i) \int_0^1 \left(\frac{1}{2}t^2 - \frac{1}{3}t^3 \right) dt = (-2+2i) \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= (-2+2i) \frac{1}{6} = -\frac{1}{3} + \frac{1}{3}i$$

c)

Vaihtoehtainen ratkaisu:

$$\oint_C (z + z^{-1}) dz = \int_0^{2\pi} (e^{it} + e^{-it}) ie^{it} dt = i \int_0^{2\pi} (e^{2it} + 1) dt$$

$$= i \int_0^{2\pi} \left(\frac{1}{2i} e^{2it} + t \right) dt = i \left(\frac{1}{2i} \underbrace{e^{i4\pi}}_{=1} + 2\pi - \frac{1}{2i} \underbrace{e^0}_{=1} - 0 \right)$$

$$= 2\pi i$$