Aalto University School of Science and Technology
Department of Mathematics and Systems Analysis

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Mat-1.3460 Principles of Functional Analysis

Exam, 23.5.2015

Exam time: 4 hours
Calculators are not allowed

1. Consider the normed space

$$c_0 = \{x = (\xi_j)_{j\in\mathbb{N}} \;:\; \xi_j \in \mathbb{C} \text{ for all } j \in \mathbb{N} \text{ and } \exists N \in \mathbb{N} \text{ so that } \xi_j = 0,\; j \geq N\}$$

with norm $\|x\| = \sup_j |\xi_j|$. Define the operator $T : c_0 \to c_0$ so that

$$(Tx)_j = \frac{1}{j!}\xi_j$$

for all $j \in \mathbb{N}$.

   (a) Show that T^{-1} is a well-defined linear operator $c_0 \to c_0$.

   (b) Prove that T is bounded, whereas T^{-1} is not.

   (c) Formulate the Open Mapping Theorem. Compare the assumptions and conclusions of the theorem to the situation of this problem.

2. Let X be a Banach space.

   (a) Give the definition of a compact set $K \subset X$.

   (b) Give the definition of a compact operator $T : X \to X$.

   (c) Show that the Hilbert cube

$$\mathcal{M} = \{x = (\xi_1, \xi_2, \ldots) \;:\; |\xi_k| \leq 2^{-k}, k \in \mathbb{N}\}$$

   is a compact set in $\ell^2(\mathbb{N})$.

3. In $\ell^2(\mathbb{N})$ we define the shift operators S and S^* by

$$Sx = (0, \xi_1, \xi_2, \ldots),$$
$$S^*x = (\xi_2, \xi_3, \ldots),$$

where $x = (\xi_1, \xi_2, \ldots)$. Show that

   (a) $\sigma_p(S) = \emptyset$ and $\sigma_p(S^*) = \{\lambda \in \mathbb{C} : |\lambda| < 1\} =: \mathbb{D}$,

   (b) $\sigma(S) = \sigma(S^*) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$,

   (c) $\partial\mathbb{D} \subset \sigma_a(S)$, $\partial\mathbb{D} \subset \sigma_a(S^*)$ and determine the corresponding approximate eigenvectors.

Please turn over

4. Define on $\ell^2(\mathbb{N})$ the mapping $P_n : (\xi_k) \mapsto (\eta_k)$ such that

$$\eta_k = \begin{cases} \xi_k, & k \leq n \\ 0, & k > n. \end{cases}$$

(a) Show that P_n is a bounded projection.

(b) Does the sequence (P_n) converge weakly, strongly or in operator norm, and if so, what is the corresponding limit operator?

5. Recall the Zorn's lemma: *Let $\mathcal{M} \neq \emptyset$ be a set with a partial ordering $\leqslant$. If every fully ordered subset has an upper bound, then $\mathcal{M}$ has a maximal element m.* Explain how this lemma has been used in the course.