Mathematics
Aalto University

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MSC-1.741 Finite Element Method

Exam 21.5.2015

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark the course code, title and text mid-term or final examination.

You have two options

1. Solve all problems. Grade is based only on the exam.
2. Solve any three problems. Grade is based on exercise points + exam points. To choose this option, you must have participated to the course during spring 2015 and completed the final project.

The exam time is three hours (3h). No electronic calculators or materials are allowed.

1. Let $a, b \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^2$ be a domain with a smooth boundary. Consider the strong problem: find u such that

$$\begin{cases} -\text{div}(A\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

in which $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

(a) Derive the weak form of the strong problem (1)

(b) Assume that $a = 1$ and $b = 0$. Use the Lax-Milgram lemma to show that there exists a unique solution to the weak problem.

(c) For which $a, b \in \mathbb{R}$ can Lax-Milgram lemma guarantee existence of a unique solution ?

Hint: For each $x \in \mathbb{R}^2$ there holds that $\lambda_{max}(A)x^t x \geq x^t Ax \geq \lambda_{min}(A)x^t x$.

2. Let the finite element mesh $\mathcal{T}$ be such that

$$p = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 5 \\ 4 & 5 & 5 & 6 \end{bmatrix} \tag{2}$$

(a) Draw the mesh $\mathcal{T}$.

(b) Compute affine mapping from the reference element to elements 3 and 4.

(c) Consider the bilinear form $a(u, v) = (\nabla u, \nabla v)$ and assume that standard linear nodal basis functions are used. Compute the row 3 of the system matrix.

3. Consider the problem : Find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V, \tag{3}$$

in which $a : V \times V \to \mathbb{R}$ is symmetric, coercive and continuous bilinear form and $L : V \to \mathbb{R}$ a continuous linear functional. Let $J : V \to \mathbb{R}$ be such that

$$J(v) = \frac{1}{2}a(v, v) - L(v)$$

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(a) Let u be a solution to Problem (3). Show that

$$J(u) < J(u+v) \quad \forall v \in V, v \neq 0.$$

(b) Let u be such that

$$J(u) = \min_{v \in V} J(v).$$

Show that u is a solution to Problem (3).

4. Let $a : V \times V \to \mathbb{R}$ be a bilinear form and $L : V \to \mathbb{R}$ a linear functional. In addition, assume that L is bounded and a is continuous, symmetric and coercive. Consider the problem : find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{4}$$

Let $V_h \subset V$. The finite element approximation to problem (2) is : find $u_h \in V_h$ such that

$$a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h.$$

(a) Formulate and prove Galerkin orthogonality property

(b) Formulate and prove Cea's Lemma