PHYS-E0415 Statistical Mechanics

Final exam, December 10, 2015, 9.00-12.00

You should answer in English unless you have special permission to use another language. You are free to use the lecture notes, the articles, the home work exercises, electronic devices, etc. Please write your name, student number, study program, course code, and the date in all of your papers. There are 3 problems in this exam set which consists of 2 pages.

Problem 1

Answer the following questions in your own words (no calculations are needed):

- According to Boltzman's definition of entropy, why can some physical systems have a negative absolute temperature?
- A different definition of entropy has been suggested by Gibbs. Describe this definition and explain why the absolute temperature is always positive according to this definition.
- Explain how Bose-Einstein condensation occurs for a free boson gas in three dimensions. Explain why Bose-Einstein condensation does not occur for free bosons in two dimensions.

Problem 2

Consider a gas of non-interacting particles (either fermions or bosons) in a two-dimensional harmonic potential. The single-particle energy states of the gas are given by

$$E_{\mathbf{n}} = \hbar\omega_0(n_x + n_y), \ \ \mathbf{n} = (n_x, n_y) \text{ with } n_x, n_y = 0, 1, 2, \ldots .$$

We ignore the zero-point energy in this problem.

a) Show that the grand partition function can be written as

$$Z = \prod_{\mathbf{n}} \left(1 + ae^{-\beta(E_{\mathbf{n}} - \mu)}\right)^a ,$$

where $\beta = 1/(k_B T)$ is the inverse temperature, μ is the chemical potential, and $a = 1$ holds for fermions and $a = -1$ for bosons.

b) For large temperatures, where the density of states for each of the two oscillator modes can be assumed to be constant, show that the Landau free energy $\phi = -k_B T \ln Z$ reads

$$\phi = a\frac{(k_B T)^3}{(\hbar\omega_0)^2} \mathrm{Li}_3(-az),$$

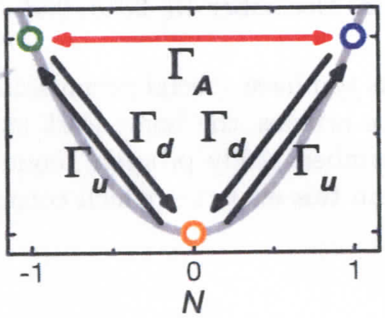
where $z = e^{\beta\mu}$ is the fugacity and $\mathrm{Li}_\nu(x) = \sum_{l=1}^{\infty} x^l/l^\nu$ is the polylogarithm of order ν. Hint: It may be useful to note that $\ln(1 + x) = -\mathrm{Li}_1(-x)$.

The exam set continues on the next page

c) Show that the internal energy is $U = -2\phi$.

d) Find the average number of particles $\langle N \rangle$.

Problem 3



The diagram above illustrates the transitions (arrows) between three states (circles) that we denote as $N = -1$, $N = 0$, and $N = 1$, respectively. Transitions from $N = 0$ to $N = \pm 1$ occur with the rate Γ_u. Transitions from $N = \pm 1$ to $N = 0$ occur with the rate Γ_d. Transitions from $N = -1$ to $N = 1$ and from $N = 1$ to $N = -1$ occur with the rate Γ_A

a) Formulate rate equations for the probabilities $P_{-1}(t)$, $P_0(t)$, and $P_1(t)$, of the system being in state $N\ (= -1, 0, 1)$ at time t.

We now consider the probabilities $P_{-1}(n,t)$, $P_0(n,t)$, and $P_1(n,t)$ that the system is in state N at time t and n transitions between $N = -1$ and $N = 1$ have occurred during the time span $[0, t]$. (We count the total number of transitions from $N = -1$ to $N = 1$ and from $N = 1$ to $N = -1$ as indicated with the double-arrow above.)

b) Formulate rate equations for the probabilities $P_{-1}(n,t)$, $P_0(n,t)$, and $P_1(n,t)$.

c) Define $P_N(\chi, t) = \sum_{n=0}^{\infty} P_N(n,t) e^{in\chi}$ with $N = -1, 0, 1$ and write down the corresponding rate equations for $P_{-1}(\chi,t)$, $P_0(\chi,t)$, and $P_1(\chi,t)$.

d) Define $P_u(\chi,t) \equiv P_{-1}(\chi,t) + P_1(\chi,t)$ and show that $P_0(\chi,t)$ and $P_u(\chi,t)$ obey a master equation of the form

$$\frac{d}{dt}\begin{pmatrix} P_u(\chi,t) \\ P_0(\chi,t) \end{pmatrix} = \begin{pmatrix} \mathcal{H}(\chi) - \Gamma_d & 2\Gamma_u \\ \Gamma_d & -2\Gamma_u \end{pmatrix} \begin{pmatrix} P_u(\chi,t) \\ P_0(\chi,t) \end{pmatrix}.$$

What is the expression for $\mathcal{H}(\chi)$ in terms of Γ_A and χ?

e) Determine the cumulant generation function of $I \equiv n/t$ for $t \to \infty$.

f) Show that the first cumulant of I reads $\langle\langle I \rangle\rangle = \Gamma_A \frac{2\Gamma_u}{2\Gamma_u + \Gamma_d}$.

g) How would you find the second cumulant of I, $\langle\langle I^2 \rangle\rangle$? (no calculations needed)

End of exam set

$$S = \frac{\mathcal{H}(\chi) - \Gamma_d - 2\Gamma_u + \sqrt{\Gamma_d^2 + 4\Gamma_u^2 + \mathcal{H}(\chi)^2 + 2\Gamma_d(-\mathcal{H}(\chi) + 2\Gamma_u) + 4\Gamma_u \mathcal{H}(\chi)}}{2}$$