Mat-2.3140 Linear Programming
Final Exam **11.05.2015**

The exam is three hours long and consists of 4 exercises. The exam is graded on a scale 0-25 points, and the points assigned to each question are indicated in parenthesis within the text.

Problem 1

(a) Solve the following problem by using the Dual Simplex method. Start with the basis defined by variables x_3, x_5, x_6. (5pt)

$$\begin{aligned}
\text{(P)}\quad \text{Minimize}\quad & -7x_1 + 7x_2 - 2x_3 - x_4 - 6x_5 \\
\text{s.t.}\quad & 3x_1 - x_2 + x_3 - 2x_4 = -3 \\
& 2x_1 + x_2 + x_4 + x_5 = 4 \\
& -x_1 + 3x_2 - 3x_4 + x_6 = 12 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

(b) Write the dual of P. Determine an optimal dual solution by using the optimal tableau of point (a). Explain how you obtain the dual solution. (2pt)

Problem 2

(a) Suppose the full tableau primal Simplex algorithm is applied to solve a minimization problem. State the unboundedness criterion for the primal Simplex algorithm, i.e., explain under which conditions it terminates with an optimal cost $-\infty$. Explain how to construct a basic direction which proves unboundedness from the final tableau. (2pt)

(b) Consider a minimization linear programming problem P and its dual D. Consider the following possibilities for P: (i) P has a finite optimal cost, (ii) P has optimal cost $-\infty$, (iii) P is infeasible. For each of the three cases indicate what are the possibilities for D. Justify your answers by using duality. (2pt)

(c) Formulate the following problem as a linear programming problem in standard form: (2pt)

$$\begin{aligned}
\text{Minimize}\quad & \max\{(-x_1 + x_2 + 2), \left(x_1 - \frac{1}{2}x_2 - \frac{1}{2}\right), (x_1 + x_2 - 1)\} \\
\text{s.t.}\quad & x_1, x_2 \geq 0.
\end{aligned}$$

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Problem 3

Consider the following optimal Simplex tableau for a minimization problem where variables x_4, x_5 are slack variables

$$\begin{aligned}
(P)\ \text{Minimize}\quad & -8x_1 + 3x_2 - 6x_3 \\
\text{s.t.}\quad & \frac{1}{8}x_2 + \frac{1}{2}x_3 \le \frac{5}{4} \\
& \frac{1}{3}x_1 - \frac{5}{24}x_2 - \frac{1}{6}x_3 \le \frac{5}{12} \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

Max $\frac{5}{4}p_1 + \frac{5}{12}p_2$

$\frac{1}{3}p_2 \le -8 \quad (x_1)$

$\frac{1}{8}p_1 - \frac{5}{24}p_2 \le 3 \quad (x_2)$

$\frac{1}{2}p_1 - \frac{1}{6}p_2 \le -6 \quad (x_3)$

$p_1, p_2 \le 0$

	$-z$	x_1	x_2	x_3	x_4	x_5
	35	0	$\frac{1}{2}$	0	20	24
x_3	$\frac{5}{2}$	0	$\frac{1}{4}$	1	2	0
x_1	$\frac{5}{2}$	1	$-\frac{1}{2}$	0	1	3

$B \quad B^{-1} \qquad I$

$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \quad \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} =$

$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

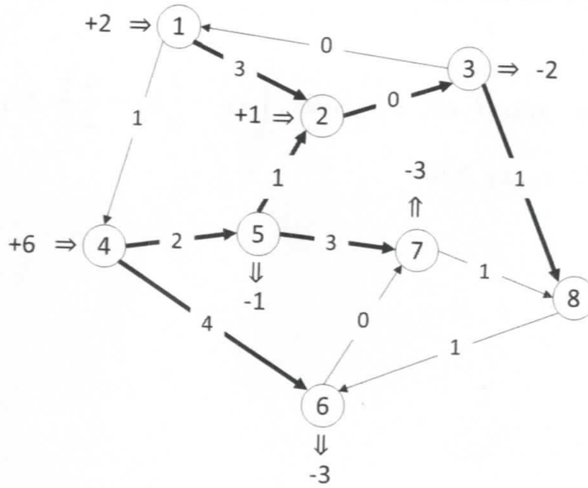
(a) Suppose $\mathbf{B}$ is the optimal basis matrix. From the tableau, determine $\mathbf{B}^{-1}$ and explain why it can be derived from the tableau (2pt)

(b) Show directly that this basis matrix $\mathbf{B}$ is optimal by deriving the corresponding primal and dual basic solutions from the tableau and then using the complementary slackness conditions (2pt)

(c) Consider adding a new variable x_4 to the original problem with coefficient 1 in both constraints and cost coefficient c_4. Determine the values of c_4 for which $\mathbf{B}$ remains optimal. (2pt)

Problem 4

Consider the uncapacitated minimum cost flow problem defined by the graph below. The number on each arc indicates its cost. Node supplies are indicated by the arrows.



$\frac{5}{2}\left(-8 - [p_1\ p_2]\begin{bmatrix} \\ \frac{1}{3} \end{bmatrix}\right)$

$\frac{5}{2}\left(-8 - \frac{1}{3}p_2\right) = 0$

$p_2 = -24$

$\frac{1}{2}p_1 - \frac{1}{6}\cdot(-24) = -6$

$p_1 - \frac{1}{3}(-24) = -12$

$p_1 + 8 = -12$

$p_1 = -20$

$\frac{1}{2}p_1 + 4 = -6$

$\frac{1}{2}p_1 = -10$

$p_1 = -20$

(a) Execute **one** iteration of the Network Simplex algorithm starting from the tree solution defined by the bold arcs (i.e., arcs $(1,2)$, $(2,3)$, $(3,8)$, $(4,5)$, $(5,2)$, $(5,7)$, $(4,6)$). Answer the following questions. (5pt)

  (i) Indicate the set T and the corresponding tree solution before and after the iteration

  (ii) Report the dual variables before and after the iteration

  (iii) Explain how the leaving and entering variables are selected and how the flows are updated

  (iv) Indicate if the tree solution obtained after the iteration can be proved optimal, and explain why

  (v) For both the primal and the dual solution obtained after the iteration indicate if the solution is degenerate and why

(b) Suppose the supply of node 6 is changed from -3 to -3+δ and that of node 8 is changed from 0 to $-\delta$ where $\delta > 0$ and small. Use the dual solution obtained at the end of point (a) to explain how the cost of the tree solution obtained at the end of point (a) changes as a function of δ. (1pt)