MS-C1080 EXAM
ALGEBRAN PERUSRAKENTEET
INTRODUCTION TO ABSTRACT ALGEBRA
23.02.2016 (3h)

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In all the assignments, a ring is assumed to have (by definition) an identity element: $1_R \in R$.
You may answer in either Finnish or English.
No calculators or tables are allowed.
You need 16p from the exam and 10p from the homework to pass the course.

1. Define/explain the following concepts:

a) (2p) Lagrange's Theorem for groups.

b) (2p) Symmetric Group $\mathbb{S}_n$.

c) (3p) Field and characteristic.

2.(6p) Let $C_\infty = \langle c \rangle$ be an infinite cyclic group. Show that if $n > 0$, then

$$C_\infty / \langle c^n \rangle \simeq C_n,$$

where C_n is a finite cyclic group with n elements.

3.(6p) Assume there are no zero divisors in the ring R. Let $a \in R$. Let n be the smallest number for which $a^n = 1$, and assume such an $n \in \mathbf{Z}_+$ exists. Assume further that n is divisible by 2. Show that

$$a^{n/2} = -1.$$

4.(6p) Prove that a subgroup of a group can be the kernel of a group homomorphism if and only if the subgroup is normal.

5. Let $\mathbb{Z}[i] = \{n + im \mid n, m \in \mathbb{Z}\}$ be the ring of Gaussian integers and let $a - ib \in \mathbb{Z}[i]$ so that $\gcd(a, b) = 1$.

a) (5p) Show that

$$\mathbb{Z}[i]/\langle a - ib \rangle \simeq \mathbb{Z}/(a^2 + b^2)\mathbb{Z}.$$

(Hint: Consider canonical projection $\pi : \mathbb{Z} \to \mathbb{Z}[i]/\langle a - ib \rangle$, $\pi(n) = n + \langle a - ib \rangle$. You can assume this map is surjective).

b) (2p) When is the quotient

$$\mathbb{Z}[i]/\langle a - ib \rangle$$

a field?

Extra You can earn +2 pts for proving that the canonical projection in a) is surjective.

Turn around for Finnish!