Kul-49.4250 Models for beam, plate and shell structures, Midterm 1

17.02.2016

1. Mapping $\vec{r}(r,\phi,z)=r^2\cos(2\phi)\vec{i}+r^2\sin(2\phi)\vec{j}+z\vec{k}$ defines a curvilinear $r\phi z-$coordinate system (differs from the cylindrical $r\phi z-$system). Use *in detail* the generic formula

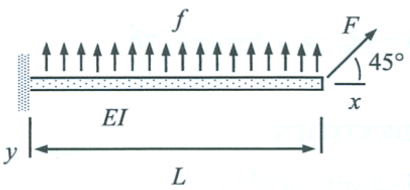
$$\frac{\partial}{\partial\alpha}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix}=(\frac{\partial}{\partial\alpha}[F])[F]^{-1}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_z\end{Bmatrix},\text{ where } \alpha\in\{r,\phi,z\}$$

to find the derivatives of the basis vectors.

2. Derive the component form of $\nabla\cdot\vec{\sigma}-\vec{f}=0$ in the polar coordinate system. Assume that the components of stress $\vec{\sigma}=\sigma_{rr}\vec{e}_r\vec{e}_r+\sigma_{r\phi}\vec{e}_r\vec{e}_\phi+\sigma_{\phi r}\vec{e}_\phi\vec{e}_r+\sigma_{\phi\phi}\vec{e}_\phi\vec{e}_\phi$ do not depend on angle ϕ. The gradient expression of the polar coordinate system and the derivatives of the basis vectors are

$$\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\phi\frac{\partial}{r\partial\phi}\,,\quad \frac{\partial}{\partial\phi}\vec{e}_r=\vec{e}_\phi \text{ and } \frac{\partial}{\partial\phi}\vec{e}_\phi=-\vec{e}_r \text{ (otherwise zeros).}$$

3. Consider the $xy-$plane beam of length L shown. Material properties E and G, cross-section properties A, I are constants, and $S=0$. Write down the boundary value problem according to the Timoshenko beam model for the axial displacement $u(x)$, transverse displacement $v(x)$, and rotation $\psi(x)$.

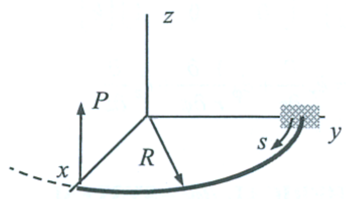


4. Virtual work expression of a torsion bar is

$$\delta W=\int_\Omega(-\delta\phi_{,x}EI_{rr}\phi_{,x}-\delta\phi k\phi+\delta\phi m)dx+(M\delta\phi)_{x=L}$$

in which $\Omega=]0,L[$ and $\phi_{,x}\equiv d\phi/dx$. Deduce *in detail* the differential equation for the rotation $\phi(x)$ and boundary conditions implied by principle of virtual work and the fundamental lemma of variation calculus. The given $EI_{rr}(x)$, $k(x)$ and $m(x)$ are *not* constants but assumed to have continuous derivatives of all orders. Rotation $\phi(0)=0$ and has continuous derivatives up to and including second order in Ω.

5. Consider the curved beam of the figure forming a 90-degree circular segment of radius R in the horizontal plane. Write down the equilibrium equations of the beam and solve for the stress resultants $N(s)$, $Q_n(s)$, $Q_b(s)$, $T(s)$, $M_n(s)$, and $M_b(s)$.



Kul-49.4250 Models for beam, plate and shell structures

INDEX NOTATION (Orthonormal basis)

$a_i b_i = \sum_{i\in I} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

$\partial a_i / \partial x_j \equiv a_{i,j}$

$\delta_{ij} \equiv \vec{e}_i \cdot \vec{e}_j \in \{0,1\} \quad (\vec{e}_i \cdot \vec{e}_j = \delta_{ij})$

$\varepsilon_{ijk} \equiv \vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k) \in \{-1,0,1\} \quad (\vec{e}_i \times \vec{e}_j = \varepsilon_{ijk} \vec{e}_k)$

$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

$\varepsilon_{ijk}\det(\mathbf{a}) = \varepsilon_{lmn} a_{il} a_{jm} a_{kn}$

GENERAL

$\vec{a} = a_i \vec{e}_i$ & $\vec{\vec{a}} = a_{ij}\vec{e}_i\vec{e}_j$ & $\vec{\vec{\vec{\vec{a}}}} = a_{ijkl}\vec{e}_i\vec{e}_j\vec{e}_k\vec{e}_l$ ...

$\vec{\vec{I}} \cdot \vec{a} = \vec{a} \cdot \vec{\vec{I}} = \vec{a} \quad \forall \vec{a} \quad (\vec{\vec{I}} = \vec{i}\vec{i} + \vec{j}\vec{j} + \vec{k}\vec{k})$

$\vec{\vec{\vec{\vec{I}}}} : \vec{\vec{a}} = \vec{\vec{a}} : \vec{\vec{\vec{\vec{I}}}} = \vec{\vec{a}} \quad \forall \vec{\vec{a}} \quad (\vec{\vec{\vec{\vec{I}}}} = \vec{i}\vec{i}\vec{i}\vec{i} + \vec{j}\vec{j}\vec{j}\vec{j} + \vec{k}\vec{k}\vec{k}\vec{k} + \vec{i}\vec{j}\vec{j}\vec{i} + \vec{j}\vec{i}\vec{i}\vec{j} + \vec{i}\vec{k}\vec{k}\vec{i} + \vec{k}\vec{i}\vec{i}\vec{k} + \vec{k}\vec{j}\vec{j}\vec{k} + \vec{j}\vec{k}\vec{k}\vec{j})$

$\vec{\vec{a}} = a_{ij}\vec{e}_i\vec{e}_j \Leftrightarrow \vec{\vec{a}}_c = a_{ij}\vec{e}_j\vec{e}_i$

$\vec{\vec{a}} = -\vec{\vec{a}}_c \Rightarrow \vec{\vec{a}} \cdot \vec{b} = \vec{a} \times \vec{b} \quad \forall \vec{b}$

IDENTITIES

$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

$\vec{\vec{a}} : (\nabla \vec{b})_c = \nabla \cdot (\vec{\vec{a}} \cdot \vec{b}) - (\nabla \cdot \vec{\vec{a}}) \cdot \vec{b}$

CYLINDRICAL $r\phi z$ **– SYSTEM**

$\vec{r} = r\cos\phi\vec{i} + r\sin\phi\vec{j} + z\vec{k}$

$$\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} \mathrm{c}\phi & \mathrm{s}\phi & 0 \\ -\mathrm{s}\phi & \mathrm{c}\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{Bmatrix} \quad \& \quad \frac{\partial}{\partial\phi}\begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_r \\ \vec{e}_\phi \\ \vec{e}_z \end{Bmatrix}$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \vec{e}_z \frac{\partial}{\partial z}$$

SPHERICAL $\theta\phi r$ **– SYSTEM**

$\vec{r}(\theta,\phi,r) = r(\mathrm{s}\theta\,\mathrm{c}\phi\vec{i} + \mathrm{s}\theta\,\mathrm{s}\phi\vec{j} + \mathrm{c}\theta\vec{k})$

$$\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{bmatrix}\mathrm{c}\theta\mathrm{c}\phi & \mathrm{c}\theta\mathrm{s}\phi & -\mathrm{s}\theta\\ -\mathrm{s}\phi & \mathrm{c}\phi & 0\\ \mathrm{s}\theta\mathrm{c}\phi & \mathrm{s}\theta\mathrm{s}\phi & \mathrm{c}\theta\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix} \ \& \ \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{Bmatrix}\mathrm{c}\theta\,\vec{e}_\phi\\ -\mathrm{s}\theta\vec{e}_r-\mathrm{c}\theta\vec{e}_\theta\\ \mathrm{s}\theta\vec{e}_\phi\end{Bmatrix} \ \& \ \frac{\partial}{\partial\theta}\begin{Bmatrix}\vec{e}_\theta\\ \vec{e}_\phi\\ \vec{e}_r\end{Bmatrix}=\begin{Bmatrix}-\vec{e}_r\\ 0\\ \vec{e}_\theta\end{Bmatrix},$$

$$\nabla=\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\vec{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}+\vec{e}_r\frac{\partial}{\partial r}$$

THIN BODY *snb* – SYSTEM FOR PLANAR BEAMS

$$\vec{r}(s,n)=\vec{r}_0(s)+n\vec{e}_n(s)$$

$$\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}\vec{r}_{0,s}/|\vec{r}_{0,s}|\\ \vec{e}_{s,s}/|\vec{e}_{s,s}|\end{Bmatrix}=\begin{Bmatrix}\vec{r}_{0,s}\\ \vec{e}_{s,s}R\end{Bmatrix} \quad \& \quad \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}\vec{e}_n/R\\ -\vec{e}_s/R\end{Bmatrix}$$

$$\nabla=\vec{e}_s\frac{R}{R-n}\frac{\partial}{\partial s}+\vec{e}_n\frac{\partial}{\partial n}$$

ORTHONORMAL CURVILINEAR COORDINATES

$$\partial_i\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}=(\partial_i[F])[F]^{-1}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}=[D]_{(i)}\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix} \ \Leftrightarrow \ \partial_i\vec{e}_j=D_{ijk}\vec{e}_k$$

$$\nabla=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}[F]^{-\mathrm{T}}[H]^{-1}\begin{Bmatrix}\partial_\alpha\\ \partial_\beta\\ \partial_n\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\alpha\\ \vec{e}_\beta\\ \vec{e}_n\end{Bmatrix}^{\mathrm{T}}[D]\begin{Bmatrix}\partial_\alpha\\ \partial_\beta\\ \partial_n\end{Bmatrix} \ \Leftrightarrow \ \nabla=\vec{e}_iD_{ij}\partial_j=\vec{e}_id_i$$

$$\Gamma_{ijk}=\vec{e}_i\cdot\nabla\vec{e}_j\cdot\vec{e}_k=\nabla\vec{e}_k=(\vec{e}_i\cdot\vec{e}_s)D_{sr}D_{rjl}(\vec{e}_l\cdot\vec{e}_k) \ \Leftrightarrow \ \Gamma_{ijk}=D_{ir}D_{rjk}$$

$$\nabla a=(d_ia)\vec{e}_i$$

$$\nabla\vec{a}=(d_ia_j+a_k\Gamma_{ikj})\vec{e}_i\vec{e}_j$$

$$\nabla\cdot\vec{a}=d_ia_i+\Gamma_{iji}a_j$$

$$\nabla\cdot\vec{\vec{a}}=(d_ia_{ij}+\Gamma_{kik}a_{ij}+\Gamma_{ikj}a_{ik})\vec{e}_j$$

$$\nabla^2a=\nabla\cdot(\nabla a)=d_id_ia+\Gamma_{jij}d_ia$$

PLATE GEOMETRY ($r\phi n$)

$$\vec{r}(r,\phi,n)=[\vec{i}r\cos\phi+\vec{j}r\sin\phi]+n\vec{e}_n$$

$$\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix} \quad \& \quad \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_r\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}\vec{e}_\phi\\ -\vec{e}_r\\ 0\end{Bmatrix}$$

$$d_r=\partial_r \quad \& \quad d_\phi=r^{-1}\partial_\phi \quad \& \quad d_n=\partial_n$$

$$\Gamma_{\phi r\phi}=-\Gamma_{\phi\phi r}=r^{-1}$$

$$dV=dnd\Omega$$

BEAM GEOMETRY (snb)

$$\vec{r}(s,n,b)=[\vec{r}_0(s)]+n\vec{e}_n+b\vec{e}_b$$

$$\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\vec{r}_{0,s}\\ \vec{e}_{s,s}/|\vec{e}_{s,s}|\\ \vec{e}_s\times\vec{e}_n\end{Bmatrix} \quad \& \quad \frac{\partial}{\partial s}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{bmatrix}0 & \kappa_b & 0\\ -\kappa_b & 0 & \kappa_s\\ 0 & -\kappa_s & 0\end{bmatrix}\begin{Bmatrix}\vec{e}_s\\ \vec{e}_n\\ \vec{e}_b\end{Bmatrix}=\begin{Bmatrix}\kappa_b\vec{e}_n\\ \kappa_s\vec{e}_b-\kappa_b\vec{e}_s\\ -\kappa_s\vec{e}_n\end{Bmatrix}$$

$$d_s=(1-n\kappa_b)^{-1}(\partial_s+\kappa_s b\partial_n-\kappa_s n\partial_b) \quad \& \quad d_n=\partial_n \quad \& \quad d_b=\partial_b$$

$$\Gamma_{ssn}=-\Gamma_{sns}=(1-n\kappa_b)^{-1}\kappa_b \quad \& \quad \Gamma_{snb}=-\Gamma_{sbn}=(1-n\kappa_b)^{-1}\kappa_s$$

$$dV=(1-n\kappa_b)dAds$$

CYLINDRICAL SHELL GEOMETRY ($z\phi n$)

$$\vec{r}(z,\phi,n)=[\vec{i}R\cos\phi+\vec{j}R\sin\phi+\vec{k}z]+n\vec{e}_n$$

$$\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{bmatrix}0 & 0 & 1\\ -\sin\phi & \cos\phi & 0\\ -\cos\phi & -\sin\phi & 0\end{bmatrix}\begin{Bmatrix}\vec{i}\\ \vec{j}\\ \vec{k}\end{Bmatrix} \quad \& \quad \frac{\partial}{\partial\phi}\begin{Bmatrix}\vec{e}_z\\ \vec{e}_\phi\\ \vec{e}_n\end{Bmatrix}=\begin{Bmatrix}0\\ \vec{e}_n\\ -\vec{e}_\phi\end{Bmatrix}$$

$$d_z=\partial_z \quad \& \quad d_\phi=(R-n)^{-1}\partial_\phi \quad \& \quad d_n=\partial_n$$

$$\Gamma_{\phi\phi n}=-\Gamma_{\phi n\phi}=(R-n)^{-1}$$

$$dV=(1-nR^{-1})dn(Rd\phi)dz=(1-nR^{-1})dnd\Omega$$

LINEAR ISOTROPIC ELASTICITY

$\vec{\vec{\sigma}}=\vec{\vec{\vec{\vec{E}}}}:\vec{\vec{\varepsilon}}=\vec{\vec{\vec{\vec{E}}}}:\nabla\vec{u}$ (minor and major symmetries of the elasticity dyad assumed)

$$\vec{\vec{\varepsilon}}=\frac{1}{2}[\nabla\vec{u}+(\nabla\vec{u})_{\mathrm{c}}]$$

$$\ddot{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} E \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix}$$

$$\ddot{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix} \text{ (plane stress)}$$

$$\ddot{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix} \text{ (beam)}$$

$$\ddot{\vec{E}} = \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{i} \\ \vec{j}\vec{j} \\ \vec{k}\vec{k} \end{Bmatrix} + \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \vec{i}\vec{j}+\vec{j}\vec{i} \\ \vec{j}\vec{k}+\vec{k}\vec{j} \\ \vec{k}\vec{i}+\vec{i}\vec{k} \end{Bmatrix} \text{ (uni-axial)}$$

$$G = \frac{E}{2(1+\nu)} \quad \& \quad D = \frac{Et^3}{12(1-\nu^2)}$$

PRINCIPLE OF VIRTUAL WORK

$$\delta W = \delta W^{\mathrm{ext}} + \delta W^{\mathrm{int}} = 0 \quad \forall \delta\vec{u} \in U \quad \text{(a function set)}$$

$$\delta W = -\int_V (\ddot{\sigma} : \delta\ddot{\varepsilon}_{\mathrm{c}})dV + \int_V (\underline{\vec{f}} \cdot \delta\vec{u})dV + \int_A (\underline{\vec{t}} \cdot \delta\vec{u})dA$$

BEAM EQUATIONS

$$\begin{Bmatrix} \vec{F}' + \underline{\vec{b}} \\ \vec{M}' + \vec{i} \times \vec{F} + \underline{\vec{c}} \end{Bmatrix} = 0 \quad \& \quad \begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix} dA$$

$$\begin{Bmatrix} \vec{F} \\ \vec{M} \end{Bmatrix} = \int \begin{Bmatrix} \vec{\sigma} \\ \vec{\rho} \times \vec{\sigma} \end{Bmatrix} dA = \int \begin{bmatrix} \ddot{E} & -\ddot{E} \times \vec{\rho} \\ \vec{\rho} \times \ddot{E} & -\vec{\rho} \times \ddot{E} \times \vec{\rho} \end{bmatrix} dA \cdot \begin{Bmatrix} \vec{u}_0' + \vec{i} \times \vec{\theta}_0 \\ \vec{\theta}_0' \end{Bmatrix} \quad \& \quad \ddot{E} = E\vec{i}\vec{i} + G\vec{j}\vec{j} + G\vec{k}\vec{k}$$

TIMOSHENKO BEAM (xyz)

$$\begin{Bmatrix} N' + \underline{b}_x \\ Q_y' + \underline{b}_y \\ Q_z' + \underline{b}_z \end{Bmatrix} = 0 \quad \& \quad \begin{Bmatrix} T' + \underline{c}_x \\ M_y' - Q_z + \underline{c}_y \\ M_z' + Q_y + \underline{c}_z \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N \\ Q_y \\ Q_z \end{Bmatrix} = \begin{Bmatrix} EAu' - ES_z\psi' + ES_y\theta' \\ GA(v'-\psi) - GS_y\phi' \\ GA(w'+\theta) + GS_z\phi' \end{Bmatrix} \quad \& \quad \begin{Bmatrix} T \\ M_y \\ M_z \end{Bmatrix} = \begin{Bmatrix} -GS_y(v'-\psi) + GS_z(w'+\theta) + GI_{rr}\phi' \\ ES_y u' - EI_{zy}\psi' + EI_{yy}\theta' \\ -ES_z u' + EI_{zz}\psi' - EI_{yz}\theta' \end{Bmatrix}$$

TIMOSHENKO BEAM (snb)

$$\begin{Bmatrix} N' - Q_n\kappa_b + \underline{b}_s \\ Q_n' + N\kappa_b - Q_b\kappa_s + \underline{b}_n \\ Q_b' + Q_n\kappa_s + \underline{b}_b \end{Bmatrix} = 0 \quad \& \quad \begin{Bmatrix} T' - M_n\kappa_b + \underline{c}_s \\ M_n' + T\kappa_b - M_b\kappa_s - Q_b + \underline{c}_n \\ M_b' + M_n\kappa_s + Q_n + \underline{c}_b \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N \\ Q_n \\ Q_b \end{Bmatrix} = \begin{Bmatrix} EA(u' - v\kappa_b) + ES_n(\theta' + \phi\kappa_b - \psi\kappa_s) - ES_b(\psi' + \theta\kappa_s) \\ GA(v' + u\kappa_b - w\kappa_s - \psi) - GS_n(\phi' - \theta\kappa_b) \\ GA(w' + v\kappa_s + \theta) + GS_b(\phi' - \theta\kappa_b) \end{Bmatrix}$$

$$\begin{Bmatrix} T \\ M_n \\ M_b \end{Bmatrix} = \begin{Bmatrix} GS_b(w' + v\kappa_s + \theta) + GI_{rr}(\phi' - \theta\kappa_b) - GS_n(v' + u\kappa_b - w\kappa_s - \psi) \\ ES_n(u' - v\kappa_b) + EI_{nn}(\theta' + \phi\kappa_b - \psi\kappa_s) - EI_{bn}(\psi' + \theta\kappa_s) \\ -ES_b(u' - v\kappa_b) - EI_{nb}(\theta' + \phi\kappa_b - \psi\kappa_s) + EI_{bb}(\psi' + \theta\kappa_s) \end{Bmatrix}$$

PLATE EQUATIONS

$$\nabla \cdot \vec{F} + \underline{\vec{b}} = 0 \quad \& \quad (\nabla \cdot \vec{M} - \vec{Q} + \underline{\vec{c}}) \times \vec{k} = 0$$

$$\vec{F} = \int \vec{\sigma} dz = \vec{ii}N_{xx} + \vec{ij}N_{xy} + \vec{ji}N_{yx} + \vec{jj}N_{yy} + (\vec{ki} + \vec{ik})Q_x + (\vec{kj} + \vec{jk})Q_y$$

$$\vec{M} = \int \vec{\sigma} z dz = \vec{ii}M_{xx} + \vec{ij}M_{xy} + \vec{ji}M_{yx} + \vec{jj}M_{yy} + (\vec{ki} + \vec{ik})R_x + (\vec{kj} + \vec{jk})R_y$$

REISSNER-MINDLIN PLATE (xyz)

$$\begin{Bmatrix} N_{xx,x} + N_{yx,y} + \underline{b}_x \\ N_{yy,y} + N_{xy,x} + \underline{b}_y \end{Bmatrix} = 0 \quad \& \quad \begin{Bmatrix} Q_{x,x} + Q_{y,y} + \underline{b}_z \\ M_{xx,x} + M_{yx,y} - Q_x + \underline{c}_x \\ M_{yy,y} + M_{xy,x} - Q_y + \underline{c}_y \end{Bmatrix} = 0 \quad \& \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gtk \begin{Bmatrix} w_{,x} + \theta \\ w_{,y} - \phi \end{Bmatrix}$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \frac{Et}{1-\nu^2} \begin{Bmatrix} u_{,x} + \nu v_{,y} \\ v_{,y} + \nu u_{,x} \\ (1-\nu)(u_{,y} + v_{,x})/2 \end{Bmatrix} \quad \& \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = D \begin{Bmatrix} \theta_{,x} - \nu\phi_{,y} \\ -\phi_{,y} + \nu\theta_{,x} \\ (1-\nu)(\theta_{,y} - \phi_{,x})/2 \end{Bmatrix}$$

$$\begin{Bmatrix} N_{nn} - \underline{N}_n & or & u_n - \underline{u}_n \\ N_{ns} - \underline{N}_s & or & u_s - \underline{u}_s \end{Bmatrix} = 0 \quad \& \quad \begin{Bmatrix} Q_n - \underline{Q}_n & or & w - \underline{w} \\ M_{ns} - \underline{M}_s & or & \theta_n - \underline{\theta}_n \\ M_{nn} - \underline{M}_n & or & \theta_s - \underline{\theta}_s \end{Bmatrix} = 0$$

KIRCHHOFF PLATE (xyz)

$$\begin{Bmatrix} N_{xx,x}+N_{yx,y}+\underline{b}_x \\ N_{yy,y}+N_{xy,x}+\underline{b}_y \end{Bmatrix}=0 \quad \& \quad \begin{Bmatrix} M_{xx,xx}+2M_{xy,xy}+M_{yy,yy}+\underline{b}_z \\ (M_{xx,x}+M_{yx,y}-Q_x+\underline{c}_x) \\ (M_{yy,y}+M_{xy,x}-Q_y+\underline{c}_y) \end{Bmatrix}=0$$

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}=\frac{Et}{1-\nu^2}\begin{Bmatrix} u_{,x}+\nu v_{,y} \\ v_{,y}+\nu u_{,x} \\ (1-\nu)(u_{,y}+v_{,x})/2 \end{Bmatrix} \quad \& \quad \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=-D\begin{Bmatrix} w_{,xx}+\nu w_{,yy} \\ w_{,yy}+\nu w_{,xx} \\ (1-\nu)w_{,xy} \end{Bmatrix}$$

$$\begin{Bmatrix} N_{nn}-\underline{N}_n \quad or \quad u_n-\underline{u}_n \\ N_{ns}-\underline{N}_s \quad or \quad u_s-\underline{u}_s \end{Bmatrix}=0 \quad \& \quad \begin{Bmatrix} Q_n+M_{ns,s}-\underline{Q}-\underline{M}_{s,s} \quad or \quad w-\underline{w} \\ M_{nn}-\underline{M}_n \quad or \quad w_{,n}+\underline{\theta}_s \end{Bmatrix}=0$$

REISSNER-MINDLIN PLATE ($r\phi z$)

$$\begin{Bmatrix} [(rN_{rr})_{,r}+N_{\phi r,\phi}-N_{\phi\phi}]/r+\underline{b}_r \\ [(rN_{r\phi})_{,r}+N_{\phi\phi,\phi}+N_{\phi r}]/r+\underline{b}_\phi \end{Bmatrix}=0 \quad \& \quad \begin{Bmatrix} N_{rr} \\ N_{\phi\phi} \\ N_{r\phi} \end{Bmatrix}=\frac{Et}{1-\nu^2}\begin{Bmatrix} u_{r,r}+\nu(u_r+u_{\phi,\phi})/r \\ \nu u_{r,r}+(u_r+u_{\phi,\phi})/r \\ (1-\nu)[(u_{r,\phi}-u_\phi)/r+u_{\phi,r}]/2 \end{Bmatrix}$$

$$\begin{Bmatrix} [(rQ_r)_{,r}+Q_{\phi,\phi}]/r+\underline{b}_z \\ [(rM_{rr})_{,r}+M_{\phi r,\phi}-M_{\phi\phi}]/r-Q_r+\underline{c}_r \\ [(rM_{r\phi})_{,r}+M_{\phi\phi,\phi}+M_{\phi r}]/r-Q_\phi+\underline{c}_\phi \end{Bmatrix}=0 \quad \& \quad \begin{Bmatrix} M_{rr} \\ M_{\phi\phi} \\ M_{r\phi} \end{Bmatrix}=D\begin{Bmatrix} \theta_{\phi,r}+\nu(\theta_\phi-\theta_{r,\phi})/r \\ \nu\theta_{\phi,r}+(\theta_\phi-\theta_{r,\phi})/r \\ (1-\nu)[(\theta_{\phi,\phi}+\theta_r)/r-\theta_{r,r}]/2 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_r \\ Q_\phi \end{Bmatrix}=Gt\begin{Bmatrix} w_{,r}+\theta_\phi \\ w_{,\phi}/r-\theta_r \end{Bmatrix}$$

ROTATION SYMMETRIC KIRCHHOFF PLATE

$$-D\nabla^2\nabla^2 w+\underline{b}_z=0 \quad \& \quad \nabla^2=\frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})$$

MEMBRANE EQUATIONS IN CYLINDRICAL GEOMETRY ($z\phi n$)

$$\begin{Bmatrix} \frac{1}{R}N_{\phi z,\phi}+N_{zz,z} \\ N_{z\phi,z}+\frac{1}{R}N_{\phi\phi,\phi} \\ \frac{1}{R}N_{\phi\phi} \end{Bmatrix}+\begin{Bmatrix} \underline{b}_z \\ \underline{b}_\phi \\ \underline{b}_n \end{Bmatrix}=0 \quad \& \quad \begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix}=\begin{Bmatrix} \frac{tE}{1-\nu^2}[u_{z,z}+\nu\frac{1}{R}(u_{\phi,\phi}-u_n)] \\ \frac{tE}{1-\nu^2}[\frac{1}{R}(u_{\phi,\phi}-u_n)+\nu u_{z,z}] \\ tG(\frac{1}{R}u_{z,\phi}+u_{\phi,z}) \end{Bmatrix}$$

MEMBRANE EQUATIONS IN SPHERICAL GEOMETRY ($\phi\theta n$)

$$\frac{1}{R}\begin{Bmatrix} \csc\theta N_{\phi\phi,\phi} + N_{\theta\phi,\theta} + \cot\theta(N_{\theta\phi} + N_{\phi\theta}) \\ \csc\theta N_{\phi\theta,\phi} + N_{\theta\theta,\theta} + \cot\theta(N_{\theta\theta} - N_{\phi\phi}) \\ N_{\phi\phi} + N_{\theta\theta} \end{Bmatrix} + \begin{Bmatrix} \underline{b}_\phi \\ \underline{b}_\theta \\ \underline{b}_n \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N_{\phi\phi} \\ N_{\theta\theta} \\ N_{\phi\theta} \end{Bmatrix} = \frac{1}{R}\begin{Bmatrix} \frac{tE}{1-\nu^2}[\csc\theta(\cos\theta u_\theta + \nu\sin\theta u_{\theta,\theta} + u_{\phi,\phi}) - (1+\nu)u_n] \\ \frac{tE}{1-\nu^2}[\csc\theta(\nu\cos\theta u_\theta + \sin\theta u_{\theta,\theta} + \nu u_{\phi,\phi}) - (1+\nu)u_n] \\ tG(\csc\theta u_{\theta,\phi} - \cot\theta u_\phi + u_{\phi,\theta}) \end{Bmatrix}$$

SHELL EQUATIONS IN CYLINDRICAL GEOMETRY ($z\phi n$)

$$\begin{Bmatrix} \kappa N_{\phi z,\phi} + N_{zz,z} + \underline{b}_z \\ N_{z\phi,z} + \kappa N_{\phi\phi,\phi} - \kappa Q_\phi + \underline{b}_\phi \\ \kappa Q_{\phi,\phi} + Q_{z,z} + \kappa N_{\phi\phi} + \underline{b}_n \end{Bmatrix} = 0 \quad \& \quad \begin{Bmatrix} M_{z\phi,z} + \kappa M_{\phi\phi,\phi} - \kappa M_{\phi n} - Q_\phi + \underline{c}_\phi \\ M_{zz,z} + \kappa M_{\phi z,\phi} - Q_z + \underline{c}_z \\ - \end{Bmatrix} = 0$$

$$\begin{Bmatrix} N_{zz} \\ N_{\phi\phi} \\ N_{z\phi} \end{Bmatrix} = \frac{Et}{1-\nu^2}\begin{Bmatrix} u_{z,z} + \nu\kappa(u_{\phi,\phi} - u_n) \\ \nu u_{z,z} + \kappa(u_{\phi,\phi} - u_n) \\ (1-\nu)(u_{\phi,z} + \kappa u_{z,\phi})/2 \end{Bmatrix} \quad \& \quad \begin{Bmatrix} M_{zz} \\ M_{\phi\phi} \\ M_{z\phi} \\ M_{\phi z} \\ M_{\phi n} \end{Bmatrix} = D\begin{Bmatrix} \omega_{z,z} + \kappa\nu\omega_{\phi,\phi} - \kappa u_{z,z} \\ \nu\omega_{z,z} + \kappa\omega_{\phi,\phi} + \kappa^2(u_{\phi,\phi} - u_n) \\ (1-\nu)(\omega_{\phi,z} + \kappa\omega_{z,\phi} - \kappa u_{\phi,z})/2 \\ (1-\nu)(\omega_{\phi,z} + \kappa\omega_{z,\phi} + \kappa^2 u_{z,\phi})/2 \\ (1-\nu)\kappa(\kappa u_{n,\phi} + \kappa u_\phi + \omega_\phi)/2 \end{Bmatrix}$$

$$\begin{Bmatrix} Q_z \\ Q_\phi \end{Bmatrix} = tG\begin{Bmatrix} u_{n,z} + \omega_z \\ \omega_\phi + \kappa(u_{n,\phi} + u_\phi) \end{Bmatrix} \quad \& \quad \begin{Bmatrix} \omega_z \\ \omega_\phi \end{Bmatrix} = \begin{Bmatrix} \theta_\phi \\ -\theta_z \end{Bmatrix}$$