MS-E2146 Integer Programming
Final Exam **06.4.2016**

The exam is three hours long and consists of 4 exercises. The exam is graded on a scale 0-65 points, and the points assigned to each exercise are indicated in parenthesis within the text.

Problem 1 (10pt)

(a) Consider the set $X = P \cap \mathbb{Z}^n$ where $P = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ and $\mathbf{A}$ is an $m \times n$ matrix with columns $\mathbf{A}_1, ..., \mathbf{A}_n$. Explain the Chvátal-Gomory procedure to derive valid inequalities for X.

(c) Consider the following Knapsack set $X = \{\mathbf{y} \in \{0,1\}^5 : 9y_1 + 8y_2 + 6y_3 + 6y_4 + 5y_5 \leq 12\}$. Provide a cover inequality for X that cuts off the point $\bar{\mathbf{x}} = (0, 1, \frac{2}{3}, 0, 0)$, and an extended cover inequality which dominates it. Explain why the extended cover inequality dominates the cover inequality.

Problem 2 (15pt)

Consider the integer knapsack problem. Given n item types, a weight a_j and a profit c_j for each item of type j, $j = 1, ..., n$, and a knapsack with capacity b, select an integer number x_j of items of each type j so that the total weight (sum of the weights of the items selected) is at most b, and the total profit of the items selected is maximum. It is assumed that a_j and b are positive integers with $a_j < b$, $\forall j = 1, ..., n$, and that $a_i \neq a_j$ if $i \neq j$, $\forall i, j = 1, ..., n$

(a) Give a dynamic programming recursion for solving the problem. Do not forget to give the initialization of the recursion.

(b) Use the recursion of point (a) to solve an instance with $n = 3$ and data $\mathbf{a} = (3, 4, 1)$, $\mathbf{c} = (7, 9, 2)$ and $b = 5$. Show the Dynamic Programming table.

Problem 3 (20pt)

Consider the following Generalized Assignment Problem

$$
\begin{aligned}
\text{(IP)} \quad \max \quad & \sum_{i \in M} \sum_{j \in N} c_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{j \in N} x_{ij} \leq 1, \quad \forall i \in M, \\
& \sum_{i \in M} a_i x_{ij} \leq b_j, \quad \forall j \in N, \\
& x_{ij} \in \{0,1\}, \quad \forall j \in N, \forall i \in M, \\
& y_j \in \{0,1\}, \quad \forall j \in N.
\end{aligned}
$$

(a) Dualize the first set of constraints and derive the corresponding Lagrangian subproblem IP($\mathbf{u}$), and the Lagrangian dual. How would you solve the Lagrangian subproblem?

(b) Let z_{LP} be the optimal cost of the LP relaxation of IP, and let w_{LD} be the optimal cost of the Lagrangian dual derived in (a). Compare z_{LP} and w_{LD}. Explain briefly your conclusion.

(c) Consider an instance of IP defined by the following data: $\mathbf{a} = (5, 8, 8, 3, 11, 4)$ and $\mathbf{b} = (12, 20, 15)$

$$(c_{ij}) = \begin{pmatrix} 6 & 2 & 1 \\ 4 & 10 & 2 \\ 3 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & 8 & 5 \\ 3 & 1 & 2 \end{pmatrix}$$

Using the Lagrangian multipliers $\bar{\mathbf{u}} = (3, 4, 2, 4, 5, 4)$ solve the Lagrangian subproblem IP($\bar{\mathbf{u}}$) derived in (a) and report its optimal solution $\mathbf{x}(\bar{\mathbf{u}})$, and the corresponding dual bound.

(d) Compute the subgradient at $\bar{\mathbf{u}}$ that would be used by the subgradient algorithm. Is it possible to conclude that $\mathbf{x}(\bar{\mathbf{u}})$ is optimal for IP using the sufficient conditions for optimality of $\mathbf{x}(\bar{\mathbf{u}})$? Why?

Problem 4 (20pt)

(a) Use the maximum cardinality matching algorithm to find a maximum cardinality matching in the bipartite graph shown in Figure 1 (see the next page). Start with the matching defined by the bold edges. (i) Show the graph with the node labels before each augmentation of the matching, (ii) report the augmenting paths found.

(b) Suppose a weight c_{ij} is associated with each edge (i, j) of the same graph and we wish to find a matching with maximum weight. Suppose the problem is formulated as an integer programming problem with variables x_{ij} for each edge (i, j). Is the polyhedron corresponding to the LP relaxation of that formulation integral (i.e., has it all integer vertices)? Justify your answer.

Kuva 1: Figure 1