Rak-54.3200 Numerical methods in structural engineering
Examination, April 7, 2016 / Niiranen

This examination consists of 4 problems, each one to be rated by following the standard scale 1...6.

Problem 1

Let us consider a long and tall wall with a constant thickness L and constant temperature values on inner and outer surfaces. A heat source inside the wall can be modelled by a quadratic function

$$f = f(x) = f_0 x(1 - x/L),$$

where x denotes the coordinate along a line across the wall and f_0 is a constant.

Accordingly, the temperature distribution inside the wall can be modelled by a one-dimensional stationary heat diffusion problem through the thickness of the wall: For given thermal conductivity k, heat source f and boundary temperature values T_0 and T_L, find the temperature distribution T such that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}x}\Big(k(x)\frac{\mathrm{d}T(x)}{\mathrm{d}x}\Big) &= f(x) \quad \forall x \in (0, L) \\ T(0) &= T_0 \\ T(L) &= T_L. \end{aligned}$$

(i) Derive the *weak form* of the *boundary value problem.*

(ii) Construct a *finite element approximation* for the problem by applying two linear elements of equal size and assuming a constant conductivity $k = k_0$ as well as zero boundary values $T_0 = 0 = T_L$.

(iii) Calculate the *finite element approximations* of the temperature and heat flow, $T_h = T_h(x)$ and

$$q_h = q_h(x) = -k(x)\frac{\mathrm{d}T_h(x)}{\mathrm{d}x},$$

respectively, at the midpoint $x = L/2$.

(iv) For a standard finite element method for the current model problem, the basic mathematical *error estimate* is of the form

$$\|T - T_h\|_1 \le Ch^k |T|_{k+1}.$$

Shortly describe the core information provided by this estimate.

Problem 2

Let us consider stationary heat conduction in the wall described in Problem 1, but now by using a two-dimensional model across the quadrangular cross section of the wall with H and L denoting the height and thickness of the wall, respectively. The bottom and top surfaces of the wall are assumed to be insulated, i.e., the heat flux across the boundary lines $y = 0$ and $y = H$, with y denoting the vertical coordinate of the cross section, is zero.

The problem can be modeled by relying on the first law of thermodynamics combined with the stationary state assumption implying the partial differential equation

$$\nabla \cdot \boldsymbol{q} = f \quad \text{in } \Omega$$

with *Fourier* law building a constitutive relation between the heat flux $\boldsymbol{q} = (q_1(x, y), q_2(x, y))$ and the temperature $T = T(x, y)$ through the thermal conductivity $k = k(x, y)$ in the form

$$\boldsymbol{q} = -k\nabla T \quad \text{in } \Omega.$$

(i) For given thermal conductivity k, heat source f and boundary temperature values T_0 and T_L, formulate the *strong form* of the two-dimensional *boundary value problem* governing the heat conduction problem described above.

(ii) Derive the *weak form* of the problem.

(iii) Describe briefly, possibly with some formulae, how a standard finite element equation system is derived from the weak form – including information about a discretization by linear quadrangular elements with *Lagrange* basis functions, as well as numerical integration and assembly of the corresponding problem vectors and matrices.

Problem 3

Let us consider an *affine mapping* of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{F}(\xi, \eta) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

(i) How does this mapping transform the origin and the two unit vectors parallel to the axes of the (ξ, η) plane? Draw a picture describing the situation, including the triangle spanned by the transformed unit vectors.

(ii) Compute the *Jacobian* matrix of the mapping and its determinant. What is the geometrical meaning of the determinant?

(iii) Explain shortly, for what purposes this kind of mappings are used in implementing finite element methods.

(iv) Explain shortly, for what purposes *mesh refinements* are used in finite element methods.

Problem 4

The bilinear form of the variational formulation corresponding to the *Reissner–Mindlin plate* bending problem can be written in the form

$$a(w,\boldsymbol{\beta};v,\boldsymbol{\eta}) = \int_\Omega \boldsymbol{D}\boldsymbol{\kappa}(\boldsymbol{\beta})\cdot\boldsymbol{\kappa}(\boldsymbol{\eta})d\Omega + \int_\Omega Gt(\nabla w-\boldsymbol{\beta})\cdot(\nabla v-\boldsymbol{\eta})d\Omega,$$

$$\boldsymbol{D} = D\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{pmatrix}, \qquad D = \frac{Et^3}{12(1-\nu^2)},$$

$$\boldsymbol{\kappa}(\boldsymbol{\beta}) = \begin{pmatrix} -\partial\beta_x/\partial x \\ -\partial\beta_y/\partial y \\ -(\partial\beta_x/\partial y + \partial\beta_y/\partial x) \end{pmatrix}.$$

(i) Define and name the quantities, variables, indeces and other notation appearing in the formulation.

(ii) Derive the corresponding bilinear form of the *variational formulation* for the *Kirchhoff–Love plate* bending problem by utilizing the bilinear form above as a starting point. Define and name the quantities, variables, indeces and other notation appearing in that formulation too.

(iii) Standard *conforming finite element formulation* for the *Reissner–Mindlin plate model* above leads to a numerical *locking* phenomena. Explain the main idea and consequences of this specific drawback of the standard formulation.