MS-E1600: Probability Theory **Exam**
Spring 2016, Aalto University **23.2.2016**

There are four problems, each worth 6 points. One problem can be compensated by exercises. For grading, that problem score will then be the integer nearest to six times the proportion of exercises you solved during the course.

Problem 1. Let (S, Σ, μ) be a measure space and $g \colon S \to [0, +\infty]$ a measurable function. For any $A \in \Sigma$, define

$$\nu[A] = \int g(s)\, 1_A(s)\, \mathrm{d}\mu(s),$$

where $1_A \colon S \to \mathbb{R}$ denotes the indicator function of the subset $A \subset S$.

(a) Show that ν is a measure on the measurable space (S, Σ). **(4 pts)**

(b) Under what assumption on the function g is the measure ν a probability measure? **(1 pts)**

(c) Interpret the probability density function of a real valued random variable as a special case of this. **(1 pts)**

Problem 2. Let $C_1, C_2, \ldots$ be independent random variables such that for any $n \in \mathbb{N}$ we have $\mathsf{P}\big[C_n = 1\big] = \frac{1}{n}$ and $\mathsf{P}\big[C_n = 0\big] = 1 - \frac{1}{n}$. Define the random variables

$$X_n = \sum_{k=0}^{n-1} 2^{-k}\, C_{n-k}.$$

(a) Show that for any $r \in \mathbb{N}$ and $n > r$ we have $\mathsf{P}\big[X_n \leq 2^{-r}\big] \geq \prod_{k=0}^{r}(1 - \frac{1}{n-k})$. Use this to show that X_n converges to 0 in probability. **(3 pts)**

(b) Show that $\mathsf{P}\big[C_n = 1 \text{ for infinitely many different } n\big] = 1$. Use this to show that X_n does not converge to 0 almost surely. **(3 pts)**

Hint: Recall one of the Borel-Cantelli lemmas.

Problem 3. Let X be a real valued random variable.

(a) Show that for any $\theta > 0$ and any $x \in \mathbb{R}$, we have **(2 pts)**

$$\mathsf{P}\big[X > x\big] \leq e^{-\theta x}\, \mathsf{E}\big[e^{\theta X}\big].$$

Suppose now that $X \sim \text{Poisson}(\lambda)$ for some parameter $\lambda > 0$, i.e., $\mathsf{P}\big[X = k\big] = \frac{\lambda^k}{k!} e^{-\lambda}$, for all $k \in \mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$, and $\mathsf{P}\big[X = k\big] = 0$ whenever $k \notin \mathbb{Z}_{\geq 0}$.

(b) Let $\theta > 0$. Calculate the expected value $\mathsf{E}\big[e^{\theta X}\big]$. **(2 pts)**

(c) Let $x > \lambda$ be fixed. Use the estimate of part (a) and the calculation in part (b) to find an upper bound for $\mathsf{P}\big[X > x\big]$. Choose the value of $\theta > 0$ which gives the optimal upper bound. **(2 pts)**

Problem 4. Let $(\Omega_1, \mathcal{F}_1, \mathsf{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathsf{P}_2)$ be two probability spaces. Consider the Cartesian product $\Omega_1 \times \Omega_2 = \big\{(\omega_1, \omega_2) \;\big|\; \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2\big\}$

(a) Define the product sigma-algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$ on $\Omega_1 \times \Omega_2$. **(1 pts)**

(b) What is the product measure $\mathsf{P}_1 \otimes \mathsf{P}_2$? Give a property which uniquely determines this measure (you do not need to prove the existence). **(1 pts)**

(c) State Fubini's theorem in this context. **(2 pts)**

Consider then the following two measure spaces (S_1, Σ_1, μ_1) and (S_2, Σ_2, μ_2). We take both $S_1 = S_2 = [0, 1]$ to be the unit interval, and $\Sigma_1 = \Sigma_2 = \mathcal{B}([0, 1])$ the Borel sigma-algebra on the unit interval. The first measure μ_1 is taken to be the ordinary Lebesgue measure on the unit interval, and the second measure μ_2 is taken to be the counting measure, $\mu_2[B] = \#B$ (number of elements in a subset $B \subset [0, 1]$). Define a function $f \colon S_1 \times S_2 \to \mathbb{R}$ by

$$f(s_1, s_2) = \begin{cases} 1 & \text{if } s_1 = s_2 \\ 0 & \text{otherwise.} \end{cases}$$

(d) Calculate both the integral $\int \Big(\int f(s_1, s_2)\, \mathrm{d}\mu_1(s_1) \Big)\, \mathrm{d}\mu_2(s_2)$ and the integral $\int \Big(\int f(s_1, s_2)\, \mathrm{d}\mu_2(s_2) \Big)\, \mathrm{d}\mu_1(s_1)$. What can you say about the validity of Fubini's theorem in this setup? **(2 pts)**