Mathematics
Aalto University

Hannukainen

MS-E1651 Numerical matrix computations

Exam 27.10.2016

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination.

You have two options

- Solve all problems. Grade is based only on the exam.
- Solve any three problems. Grade is based on exercise points + exam points.

The exam time is three hours (3h). No electronic calculators or materials are allowed.

1. Let $X \in \mathbb{R}^{n \times n}$ such that $X = X^T$ and $\| \cdot \|$ be an operator norm induced by the Euclidian norm $\| \cdot \|_2$. Show that

   (a)
   $$\|X\| = |\lambda_{max}(X)| \quad \text{and} \quad \|Q^T X U\| = \|X\|,$$
   in which λ_{max} is the largest eigenvalue of X and $Q, U \in \mathbb{R}^{n \times n}$ are unitary matrices.

   (b) when $\|X\| < 1$ there holds that
   $$\|(I - X)^{-1}\| = \frac{1}{1 - \|X\|}.$$

2. Let $\boldsymbol{b} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite. Consider minimising the functional $J : \mathbb{R}^n \to \mathbb{R}$,
   $$J(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{x}$$
   using the line search method starting from initial guess $\boldsymbol{x}_0$. Denote iterates as $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ and the search direction on step n as $\boldsymbol{p}_n$.

   (a) Derive the formula for computing $\boldsymbol{x}_n$ from $\boldsymbol{x}_{n-1}$

   (b) Let $\boldsymbol{x}_0$ be the initial guess and $\boldsymbol{p}_1, \boldsymbol{p}_2$ be A-orthogonal search directions on steps $n = 1$ and $n = 2$. Show that $\boldsymbol{x}_2$ satisfies
   $$J(\boldsymbol{x}_2) = \min_{\alpha, \beta \in \mathbb{R}} J(\boldsymbol{x}_0 + \alpha \boldsymbol{p}_1 + \beta \boldsymbol{p}_2)$$

3. Consider the problem: Given $A \in \mathbb{C}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{C}^m$, find $\boldsymbol{x} \in \mathbb{C}^n$ such that

$$\|A\boldsymbol{x} - \boldsymbol{b}\|_2 \tag{1}$$

is minimized. Assume, that A is of full rank.

(a) Find the Givens-rotation matrix $Q_x \in \mathbb{R}^{2\times 2}$ such that

$$Q_x\boldsymbol{x} = \begin{bmatrix} \|\boldsymbol{x}\|_2 \\ 0 \end{bmatrix}$$

(b) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \pi \\ 1 & 0 \end{bmatrix}$$

Compute the QR-decomposition of A using Givens rotation matrices.

(c) Derive a formula for solving the least squares problem (1) using QR-decomposition.

4. Let $A \in \mathbb{R}^{n\times n}$ be such that

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix},$$

in which $A_{11} \in \mathbb{R}^{k\times k}$, $k < n$. In addition, let $A_{11} = L_{11}L_{11}^T$ and $A_{22} = L_{22}L_{22}^T$, in which L_{11} and L_{22} are invertible lower triangular matrices.

(a) Show that A is positive definite.

(b) Give the Cholesky decomposition of A.

(c) Let $\boldsymbol{b} \in \mathbb{C}^n$. Explain, how Cholesky-decomposition of A can be used to solve the problem $A\boldsymbol{x} = \boldsymbol{b}$.