Exam Nanoelectronics: Dec 21, 2016

1. *Basic concepts*

- *Tunneling and Coulomb blockade:* Explain briefly the phenomenon of Coulomb blockade. What is the relevance of temperature in this phenomenon? In which devices is Coulomb blockade used? What is inelastic and elastic co-tunneling?
- *Density of states:* Does the density of states depend on dimensionality (e.g. 1D, 2D, 3D) of the sample or not? What features are distinct in the density of states of a superconductor in comparison to a normal metal?
- *Qubits:* Briefly describe the physical process of Rabi oscillations in the on-resonance case.
- *Noise and correlations:* Briefly describe the type of noises commonly seen in mesoscopic physics. What is the Fano factor? What is the distribution of shot noise power spectrum of completely uncorrelated particles? What happens when particles are correlated?
- *Amplification:* Discuss briefly the Haus-Caves theorem.

2. *Semiclassical transport theory*

The Boltzmann equation in the diffusive regime can be written

$$(\partial_t + \vec{v}\cdot\nabla)[f_0(\vec{r},E,t) + \delta\vec{f}(\vec{r},E,t)\cdot\hat{\vec{p}}] = -\frac{1}{\tau}\delta\vec{f}(\vec{r},E,t)\cdot\hat{\vec{p}}. \tag{1}$$

(a) What is meant by saying a system is in the diffusive regime? How does it differ from the ballistic regime?

(b) Assuming that $\delta\vec{f}$ varies slowly in time, derive diffusion equations for f_0 for both one and two spatial dimensions. Compare the respective diffusion constants.

3. *Landauer's current formula*

A nanowire is attached to left and right reservoirs, and the current propagates from the left reservoir to the right reservoir (in the $+z$ direction). The wave function in the αth reservoir ($\alpha = L, R$) can be expressed with the summing over incoming and out-going modes. In the energy representation in the second quantized picture wave function can be expressed in the following form of field operator,

$$\hat{\Psi}_\alpha(\vec{r},t) = \int dE e^{-iEt/\hbar}\sum_{n=1}^{N_\alpha(E)} \frac{\chi_{\alpha n}(x,y)}{\sqrt{2\pi\hbar v_{\alpha n}(E)}}\left[\hat{a}_{\alpha n}(E)e^{ik_{\alpha n}(E)z} + \hat{b}_{\alpha n}(E)e^{-ik_{\alpha n}(E)z}\right], \tag{2}$$

where $\chi_n(x,y)$ is the eigenfunction of the transverse mode in $x-y$ plane (think about particles in a box), and transverse wave function has the orthogonality property that $\int dxdy\chi^*_{\alpha n}\chi_{\alpha m} = \delta_{mn}$

From quantum mechanics, we know that the current operator in the αth reservoir is given by:

$$\hat{I}_\alpha(z,t) = \frac{e\hbar}{2im}\int dxdy\left[\hat{\Psi}^\dagger_\alpha(\vec{r},t)\left(\frac{\partial}{\partial z}\hat{\Psi}_\alpha(\vec{r},t)\right) - \left(\frac{\partial}{\partial z}\hat{\Psi}^\dagger_\alpha(\vec{r},t)\right)\hat{\Psi}_\alpha(\vec{r},t)\right]. \tag{3}$$

(a) Assuming that the quasi-momenta $k_{\alpha n}(E), k_{\alpha n}(E')$ of particles with energy E, E' near Fermi level are weakly dependent on energy, i.e. $k_{\alpha n}(E) - k_{\alpha n}(E') \approx 0$, and the velocity is related to quasi-momentum as $v_{\alpha n}(E) = \hbar k_{\alpha n}(E)/m$. Find the current operator in terms of operators of $\hat{a}_{\alpha n}, \hat{a}^\dagger_{\alpha n}, \hat{b}_{\alpha n}, \hat{b}^\dagger_{\alpha n}$

(b) Rewrite the current operator in terms only in $\hat{a}_{\alpha n}, \hat{a}^\dagger_{\alpha n}$, using following scattering matrix operators

$$\hat{b}_{\alpha n}(E) = \sum_{\beta,m} s^{\alpha\beta}_{nm}\hat{a}_{\beta m}(E), \tag{4}$$

$$\hat{b}^\dagger_{\alpha n}(E) = \sum_{\beta,m} (s^{\alpha\beta}_{nm})^*\hat{a}^\dagger_{\beta m}(E). \tag{5}$$

4. *Graphene*

In one of the exercise sets, we have solved Haldane's model of graphene, but we only took into account the situation with real next nearest neighbour (NNN) hopping. Now we consider a complex NNN hopping (this is the original Haldane's model), with $t' \to t'e^{-i\phi}$ for the A atoms and $t' \to t'e^{i\phi}$ for the B atoms,

$$H = H_{\mathrm{NN}} + H_{\mathrm{NNN}} \tag{6}$$

$$= -t \sum_{<i,j>} \Psi_i^\dagger \Psi_j - t'e^{-i\phi} \sum_{<<i,j>>}^{A} \Psi_i^\dagger \Psi_j - t'e^{i\phi} \sum_{<<i,j>>}^{B} \Psi_i^\dagger \Psi_j + h.c. \tag{7}$$

where $< i,j >$ denotes the sum over nearest neighbour (NN) hopping (this term corresponds to the Hamiltonian discussed in the lecture), $<< i,j >>$ denotes the sum over next nearest neighbour hopping and t and t' are real hopping strengths. Note that the next nearest neighbor hopping can occur on either the A lattice or B lattice separately, therefore we use a superscript A and B for the corresponding sums in the Hamiltonian. Thus, in the next nearest neighbor Hamiltonian an electron from an A atom can hop only to an A atom with complex hopping amplitude $t'e^{-i\phi}$, while when electrons hop between B atoms the complex hopping amplitude is $t'e^{i\phi}$.

(a) Define the vectors in lattice space of the NN and NNN hopping displacements, and write down the Hamiltonian specifically term by term.

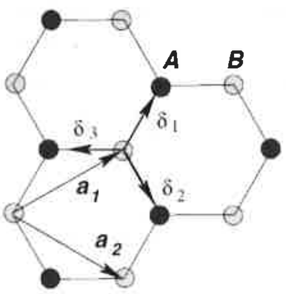
(b) Perform discrete Fourier transformations just like the procedure in the graphene lecture, and transform the Hamiltonian into k-space. In the k-space, ignoring for simplicity the spin index, we have $c_{A,\vec{k}} = \frac{1}{\sqrt{N}} \sum_{\vec{R}_i} e^{i\vec{k}\vec{R}_i} \hat{\Psi}_A(\vec{R}_i)$ and similarly for the B atoms.

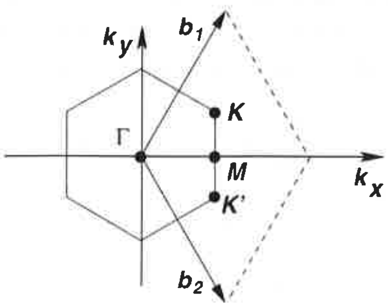
(c) Rewrite the k-space Hamiltonian the following form, and identify the $h_{\vec{k}}$ matrix.

$$H = \sum_{\vec{k}} \begin{pmatrix} \hat{c}^\dagger_{B,\vec{k}} & \hat{c}^\dagger_{A,\vec{k}} \end{pmatrix} h_{\vec{k}} \begin{pmatrix} \hat{c}_{B,\vec{k}} \\ \hat{c}_{A,\vec{k}} \end{pmatrix}. \tag{8}$$

(d) Rewrite $h_{\vec{k}} = d_0(\vec{k})I + \vec{d}(\vec{k}) \cdot \vec{\sigma}$ where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the Pauli matrix vector of different valleys. Calculate the energy spectrum $E_\pm(\vec{k})$, either directly by diagonalization or by using the property of Pauli matrices that $(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b})I + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}$.

(e) Discuss the energy gap $E_{\mathrm{gap}}(\vec{k}) = E_+(\vec{k}) - E_-(\vec{k})$ at the K and K' points in the following two situations: (i) Without NNN hopping ($t' = 0$) (ii) with NNN hopping ($t' \neq 0$).





You might find useful the following information:

In real lattice space, we define the following lattice vectors:

$$\vec{\delta}_1 = \frac{a}{2}\vec{e}_x + \frac{\sqrt{3}}{2}a\vec{e}_y, \tag{9}$$
$$\vec{\delta}_2 = \frac{a}{2}\vec{e}_x - \frac{\sqrt{3}}{2}a\vec{e}_y, \tag{10}$$
$$\vec{\delta}_3 = -a\vec{e}_x + 0\vec{e}_y, \tag{11}$$
$$\vec{a}_1 = \frac{3a}{2}\vec{e}_x + \frac{\sqrt{3}}{2}a\vec{e}_y, \tag{12}$$
$$\vec{a}_2 = \frac{3a}{2}\vec{e}_x - \frac{\sqrt{3}}{2}a\vec{e}_y, \tag{13}$$
$$\vec{a}_3 = 0\vec{e}_x + \sqrt{3}a\vec{e}_y = \vec{a}_1 - \vec{a}_2, \tag{14}$$
$$\vec{K} = \frac{2\pi}{3a}(\vec{e}_x + \frac{1}{\sqrt{3}}\vec{e}_y), \tag{15}$$
$$\vec{K'} = \frac{2\pi}{3a}(\vec{e}_x - \frac{1}{\sqrt{3}}\vec{e}_y). \tag{16}$$
$$\tag{17}$$

The Pauli matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{18}$$

5. *Input-output theory*

Consider a system consisting of a *double-sided* cavity, i.e, a cavity coupled to two separate input-output modes. The equation of motion for the cavity operator a is then

$$\dot{\hat{a}}(t) = -i\omega_c\hat{a}(t) - \left(\frac{\kappa_1}{2} + \frac{\kappa_2}{2}\right)\hat{a}(t) - \sqrt{\kappa_1}\hat{b}_{1,\text{in}} - \sqrt{\kappa_2}\hat{b}_{2,\text{in}}.$$

(a) Briefly discuss possible interpretations of each of the terms in the equation of motion. Use the equation to find an expression for $\hat{a}[\omega]$.

(b) Using standard input-output relations ($\hat{b}_{1,2,\text{out}} = \hat{b}_{1,2,\text{in}} + \sqrt{\kappa_{1,2}}\hat{a}$), express the output modes $\hat{b}_{1,\text{out}}$, $\hat{b}_{2,\text{out}}$ in terms of the input modes.

(c) The equation for $\hat{b}_{1,\text{out}}$ is of the form $\hat{b}_{1,\text{out}} = c_1\hat{b}_{1,\text{in}} + c_2\hat{b}_{2,\text{in}}$, where c_1, c_2 are complex numbers. What do these numbers represent? Calculate $|c_1|^2 + |c_2|^2$ and explain the result briefly.

(d) Assume that the system is very near resonance. Interpret physically the equations for $\hat{b}_{1,\text{out}}$ and $\hat{b}_{2,\text{out}}$ in the case $\kappa_1 = \kappa_2$ as well as $\kappa_1 \gg \kappa_2$. In the latter case, what happens away from resonance?