MS-E1600 Probability Theory
Department of Mathematics and Systems Analysis
Aalto University

Exam
4.4.2017, 13:00–16:00
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You are allowed to bring to the exam a handwritten **memory aid sheet**. The memory aid sheet must be of size A4 with text only on one side and it must contain your name and student number in the upper right corner. You don't need to return your memory aid sheet. The exam consists of 4 problems, each worth 6 points.

The grading of the course is based on either 100% exam score or 50% exam score + 40% homeworks score + 10% quizzes score, whichever is better.

1. Let $c \in \mathbb{R}$. For $n \in \mathbb{N}$ define the function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \frac{1}{1+e^{n(x-c)}}$.

   (a) Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges pointwise, and calculate the limit function. (2 p)

   (b) Suppose that X is a real valued random variable whose cumulative distribution function $x \mapsto F_X(x) = \mathsf{P}[X \leq x]$ is continuous. Show that then we have $\mathsf{E}[f_n(X)] \to F_X(c)$ as $n \to \infty$. (2 p)

   (c) Do we have $\mathsf{E}[f_n(X)] \to F_X(c)$ as $n \to \infty$ for all real valued random variables X? Provide a proof or give a counterexample. (2 p)

2. This problem concerns properties and examples of sigma algebras.

   (a) Let S be a set. Give the defining properties for a collection Σ of subsets of S to be a sigma algebra on S. (1 p)

   (b) Consider $S = \mathbb{R}$. Let Σ_1 be the collection of all subsets of $\mathbb{R}$ and let Σ_2 be the collection of all open subsets of $\mathbb{R}$. Are Σ_1 and Σ_2 sigma algebras on $\mathbb{R}$? (1 p)

   Let I be an index set and assume that for all $i \in I$ we have a sigma algebra Σ_i on S.

   (c) Is the intersection $\bigcap_{i\in I} \Sigma_i$ a sigma algebra on S? Provide a proof or give a counterexample. (2 p)

   (d) Is the union $\bigcup_{i\in I} \Sigma_i$ a sigma algebra on S? Provide a proof or give a counterexample. (2 p)

3. Let X be a non-negative random variable. Recall that for a given $p \geq 1$ we say that X is p-integrable and write $X \in \mathcal{L}^p$ if and only if the random variable X^p is integrable. Let $F(x) = \mathsf{P}[X \leq x]$ be the cumulative distribution function of X.

   (a) Prove that for $p \geq 1$ we have **(3 p)**

$$\mathsf{E}[X^p] = p \int_0^\infty s^{p-1}\,(1 - F(s))\,\mathrm{d}s.$$

   Hint: *Note that for $x \geq 0$ one has*

$$x^p = \int_0^x \left(\frac{\mathrm{d}}{\mathrm{d}s} s^p\right) \mathrm{d}s.$$

   Assume that for some $\alpha > 0$ the cumulative distribution function of X satisfies

$$\lim_{x \to \infty} \Big(x^\alpha\,(1 - F(x))\Big) = c > 0.$$

   (b) Prove that $X \in \mathcal{L}^p$ if and only if $p < \alpha$. **(3 p)**

4. Let $X_3, X_4, \ldots$ be independent random variables such that for $k = 3, 4, \ldots$ we have

$$\mathsf{P}\big[X_k = 0\big] = 1 - \frac{1}{k\,\log(k)} \qquad \text{and} \qquad \mathsf{P}\big[X_k = +k\big] = \frac{1}{2k\,\log(k)} = \mathsf{P}\big[X_k = -k\big].$$

   (a) Calculate the expected value and variance of X_k. **(1 p)**

   (b) Show that we have **(1 p)**

$$\sum_{j=3}^\infty \frac{1}{j\,\log(j)} = \infty \qquad \text{and} \qquad \frac{1}{n^2} \sum_{j=3}^n \frac{j}{\log(j)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

   Hint: *Recall that the integral function of $x \mapsto \frac{1}{x\,\log(x)}$ is $x \mapsto \log(\log(x))$. Consider breaking the second sum into two pieces, $j \leq b_n$ and $j > b_n$, with a judiciously chosen b_n, so that you can easily estimate the pieces.*

   For $n \geq 3$, define the average

$$A_n = \frac{1}{n-2} \sum_{k=3}^n X_k.$$

   (c) Does the sequence $(A_n)_{n=3,4,\ldots}$ of averages converge almost surely? **(2 p)**

   (d) Does the sequence $(A_n)_{n=3,4,\ldots}$ of averages converge in probability? **(2 p)**