Aalto University
Department of Information and Computer Science

T-79.5205 Combinatorics (5 cr)
Exam Mon 16 Dec 2013, 9–12 a.m.

Write down on each answer sheet:
- Your name, degree programme, and student number
- The text: "T-79.5205 Combinatorics 16.12.2013"
- The total number of answer sheets you are submitting for grading

1. An *involution* of $[n] = \{1,2,\ldots,n\}$ is a permutation σ of $[n]$ that is its own inverse, that is, $\sigma^{-1} = \sigma$.

   (a) List all involutions of $[n]$ for $n = 2,3,4$.

   (b) Denote by $I(n)$ the number of involutions of $[n]$. Determine $I(n)$ for $n = 5,6,7$. *Hint:* Consider the decomposition of σ into cycles. You may want to derive a recurrence relation or a direct counting formula. Make sure your formula agrees with the lists in part (a)!

2. The principle of inclusion and exclusion.

   (a) Give a careful description of the principle of inclusion and exclusion.

   (b) How many positive integers at most 1000 are not divisible by any of the integers $2,3,4,5,6$?

3. Partially ordered sets.

   (a) Show that the set of positive integer divisors of the integer n is partially ordered by the integer divisibility relation "$|$", where $a|b$ holds for integers a and b if and only if there exists an integer q such that $qa = b$.

   (b) Draw a Hasse diagram of the positive integer divisors of $n = 60$. Find a largest antichain and a partition of the divisors into the minimum possible number of chains.

4. Combinatorial and probabilistic proof techniques.

   (a) For $n \geq 1$, let $A_1, A_2, \ldots, A_k \subseteq [n]$ be distinct sets with $A_i \cap A_j \neq \emptyset$ for all $1 \leq i < j \leq k$. Prove that $k \leq 2^{n-1}$ and that equality can hold.
   Hint: Start with an explicit construction for equality.

   (b) Let n and s be nonnegative integers such that $n \geq 3$ and $2s \log 2 > 3 \log n$. Prove that there exist sets $A_1, A_2, \ldots, A_s \subseteq [n]$ such that for every $B \in \binom{[n]}{3}$ there exists a $1 \leq j \leq s$ with both $A_j \cap B \neq \emptyset$ and $A_j \cap B \neq B$.
   Hint: Use a nonconstructive technique.

Grading: Each problem 12p, total 48p.