Aalto University **Department of Information and Computer Science**

T-79.5205 Combinatorics (5 cr)
Complete Course Exam and/or **II-nd Midterm Exam Mon 19 May 2014**

Write down on each answer sheet:
- Your name, degree programme, and student number
- The text: "T-79.5205 Combinatorics 19.05.2014"
- The total number of answer sheets you are submitting for grading

Note: If you have completed/passed the Midterm Exam from 18.02.2014 then (**and only then!**) you can take the "II-nd Midterm Exam", i.e. problems 5 to 8. Otherwise you have to solve the "Complete Course Exam", i.e. problems 1 to 4.

Complete Course Exam

1. Give a closed-form solution or a counting recurrence as a function of n and k.

   (a) Determine the number of binary strings of length n that do not contain a substring of the form "11". For example, for $n = 6$ the binary string 100101 is such a string, whereas 101100 is not. Compute the values for $1 \leq n \leq 10$.

   (b) As in part (a), but with the additional constraint that the number of "1"s in the string is exactly k. Compute the value for $n = 6$ and $0 \leq k \leq 6$.

   Hint: Start by listing the strings for small values of n.

2. (a) Let $[n] = \{1,2,\ldots,n\}$ and denote by Π_n the set of all set partitions of $[n]$. For set partitions $\sigma = \{S_1,S_2,\ldots,S_p\}$ and $\tau = \{T_1,T_2,\ldots,T_q\}$ of $[n]$, define $\sigma \leq \tau$ if and only if for every $i = 1,2,\ldots,p$ there exists a $j = 1,2,\ldots,q$ with $S_i \subseteq T_j$. Show that Π_n is partially ordered by $\leq$.

   (b) A chain in a partially ordered set is *maximal* if the chain is not a proper subset of a larger chain. For $n = 6$, give an example of a maximal chain in $(\Pi_n, \leq)$.

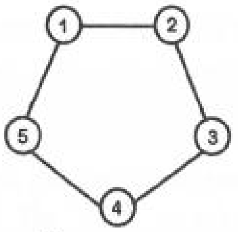
   (c) Derive an expression for the number of maximal chains in $(\Pi_n, \leq)$.

3. (a) For $n \geq 1$, let $A_1, A_2, \ldots, A_k \subseteq [n]$ be distinct sets with $A_i \cap A_j \neq \emptyset$ for all $1 \leq i < j \leq k$. Prove that $k \leq 2^{n-1}$ and that equality can hold.
   Hint: Start with an explicit construction for equality.

   (b) Provide an upper bound for the Ramsey number $R(3,4)$. That is, find an N such that for all $n \geq N$ a graph with n nodes contains either a clique of size 3 or an independent set of size 4 (or both). Use either Ramsey's Theorem for Graphs, in which case you have to prove it, or provide a constructive proof.
   Hint: For example, you can generalize the constructive proof for $R(3,3) \geq 6$.

4. Consider the 5-cycle graph C_5 drawn below.

   (a) Determine the elements of the automorphism group $\mathrm{Aut}(C_5) \leq S_5$.



(b) Draw the orbits of $\mathrm{Aut}(C_5)$ on $\binom{[5]}{2}$.[1]

(c) Give an example of a graph with at least two vertices whose only automorphism is the identity permutation. Carefully justify why this is the case.
Hint: You will need at least six vertices.

Grading: Each problem 12p, total 48p.

II-nd Midterm Exam

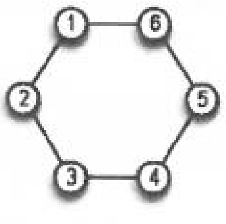
5. (a) What is a Steiner Triple System of order v?

   (b) Let $X = \mathbb{Z}_{13}$, and let $B_1 = \{0,1,4\}$ and $B_2 = \{0,2,8\}$. Let $\mathcal{B} = \{B_1 + z, B_2 + z : z \in X\}$. Prove that the pair $(X, \mathcal{B})$ forms a *Steiner triple system* of order 13. (Hint: Note that for any nonzero $z \in X$, there is a unique way to write z as $z = x - y$ with $x, y \in B_i$)

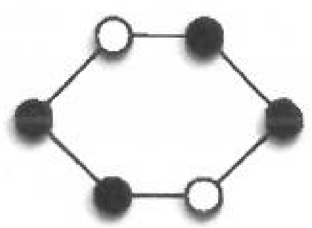
6. (a) What is a *sunflower* with *k petals*? Prove that if $\mathcal{F}$ is an s-uniform set family with $|\mathcal{F}| > s!(k-1)^s$, then $\mathcal{F}$ contains a sunflower with k petals (Sunflower Lemma).

   (b) Show that any finite graph contains two vertices lying on the same number of edges.

7. (a) Let us study the 6-cycle C_6 below. Determine the elements of the automorphism group $\mathrm{Aut}(C_6) \leq S_6$.



   (b) A necklace is constructed from 6 pearls, where each pearl is either black or white. For example, one possible necklace with 2 white pearls and 4 black pearls is depicted below.



[1] We write $\binom{[n]}{k}$ for the set of all k-subsets of $[n] = \{1,2,\dots,n\}$. A permutation σ of $[n]$ acts on $S \subseteq [n]$ by $\sigma(S) = \{\sigma(x) : x \in S\}$.

Using the Orbit Counting Lemma, count the number of distinct necklaces.

Hint: The distinct necklaces can be viewed as the orbits of the automorphism group $Aut(C_6)$ on $2^{[6]}$.

8. (a) An automatic machine assigns pre-produced (and unique) name tags to the n participants of a conference. The machine gets broken and now assigns the name tags independently and uniformly at random (several tags may end up to the same person). What is the expected number of correct matches?

   (b) A fully mechanized (and highly complex) machine receives as input a set W of binary strings of size n, and, according to a secret assignment, for each word $w \in W$ selects k string positions, not necessarily in order, and outputs the k-bit projection corresponding to these positions. The machine gets broken, and as a result it will always select the same k sting positions independent of the input. Knowing about the malfunction but not knowing about which k positions the machine selects, an operator wants to design a set W of n-bit inputs such that (independent of which k positions the machine got stuck into) the collection of all outputted words contains all 2^k possible binary strings of length k. Show that if $|W|$ obeys the following inequality, it is always possible to design such an input set W:

$$\binom{n}{k} 2^k (1 - 2^{-k})^{|W|} < 1.$$

   Hint: *Use the counting sieve.*

Grading: Each problem 6p, total 24p.