Mathematics
Aalto University

Hannukainen / Gustafsson

MSC-1.741 Finite Element Method

Exam 4.4.2017

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark the course code, title and text mid-term or final examination.

You have two options

1. Solve all problems. Grade is based only on the exam.
2. Solve any three problems. Grade is based on exercise points + exam points. To choose this option, you must have completed the final project.

The exam time is three hours (3h). No electronic calculators or materials are allowed.

1. Let Ω be a bounded domain and $f \in L^2(\Omega)$. Consider the strong problem: find u such that

$$\begin{cases} -\nabla \cdot (\nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1}$$

   (a) Derive the weak form of the strong problem (1) : find $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \in V.$$

   Specify a, L, the space V and it's norm $\|\cdot\|_V$.

   (b) Show that there exists $C_1, C_2 > 0$ independent on u and v such that

$$|a(u,v)| \leq C_1 \|u\|_V \|v\|_V \quad \text{and} \quad |L(v)| \leq C_2 \|v\|_V \quad \forall u, v \in V.$$

2. Let the reference element be $\hat{K} = (0,0), (1,0), (0,1)$ and the reference basis functions

$$\hat{\varphi}_1(\hat{x},\hat{y}) = 1 - \hat{x} - \hat{y} \quad \hat{\varphi}_2(\hat{x},\hat{y}) = \hat{x} \quad \text{and} \quad \hat{\varphi}_3(\hat{x},\hat{y}) = \hat{y}$$

   Consider the element $K = (\frac{1}{2}, 0), (\frac{3}{2}, 2), (\frac{1}{2}, 1)$.

   (a) Compute the affine mapping from $\hat{K}$ to K

   (b) Give the definition of first order nodal based functions on element K. How are these functions related to reference basis functions ?

   (c) Compute the gradient of each of the global basis functions.

   (d) Compute the entry $(1,3)$ of the local stiffness matrix on element K related to the bilinear form $(\nabla u, \nabla v)$.

   Hint: Inverse of any 2×2-matrix can be computed as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}. \tag{2}$$

3. Let $a : V \times V \to \mathbb{R}$ be a bilinear form and $L : V \to \mathbb{R}$ a linear functional. In addition, assume that L is bounded and a is continuous as well as elliptic. Consider the problem : find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{3}$$

Let $V_h \subset V$ and $u_h \in V_h$ be such that

$$a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h.$$

(a) Formulate and prove Galerkin orthogonality property

(b) Formulate and prove Cea's Lemma

(c) Explain how Cea's lemma can be used to derive finite element error estimates.

4. Let $a : V \times V \to \mathbb{R}$ be a bilinear form and $L : V \to \mathbb{R}$ a linear functional. In addition, assume that L is bounded and a is continuous as well as elliptic.

(a) Let u be a solution to the problem: find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{4}$$

Show that u the global minimiser of

$$J(u) = \frac{1}{2}a(u, u) - L(u).$$

(b) Let u be the global minimiser of

$$J(u) = \frac{1}{2}a(u, u) - L(u).$$

Show that u is a solution to : find $u \in V$ such that

$$a(u, v) = L(v) \quad \forall v \in V. \tag{5}$$