Aalto University
School of Science

PHYS-E0416 Quantum Physics
2nd exam 5.4.2017

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1. Write a mini-essay (0.5-1 page each) about the following two topics:
   a) Rabi oscillations, Rabi splitting and avoided crossing. No derivations are needed, just use words, pictures and simple equations.
   b) BCS-BEC crossover. For full points it is preferable to write down the BCS and BEC wavefunctions. Otherwise equations are not necessary, you can explain things by words, and possibly pictures if you wish.

2. a) Show that an arbitrary qubit gate can be decomposed (up to a phase factor) into three consecutive rotations around two orthogonal axes. In other words, if U is a 2×2 unitary transformation, prove that there exist four real numbers α, β, γ and δ such that

   $$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

   The rotations are defined as $R_w(\varphi) = e^{-\frac{i}{2}\varphi\sigma_w}$. Could this result be helpful in any way for the implementation of a single-qubit gate in a real system?

   b) Find α, β, γ and δ for the Hadamard gate.

   Pauli spin-matrices were

   $$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

3. To describe e.g. bosonic atoms in an optical lattice one may use the Bose-Hubbard Hamiltonian

   $$H = -J \sum_{<i,j>} a_i^\dagger a_j + \frac{U}{2} \sum_i a_i^\dagger a_i^\dagger a_i a_i$$

   together with the Gutzwiller mean-field ansatz

   $$|\Psi_{MF}\rangle = \prod_{i=1}^{M} \left[\sum_{n_i=0}^{\infty} f_{n_i}^{(i)} |n_i\rangle \right].$$

   a) Calculate the following quantities:

   $$\langle \Psi_{MF}|H|\Psi_{MF}\rangle,$$
   $$\langle \Psi_{MF}|\hat{n}_i|\Psi_{MF}\rangle,$$
   $$\langle \Psi_{MF}|a_i|\Psi_{MF}\rangle,$$
   $$\sigma_i^2 = \frac{\langle \Psi_{MF}|\hat{n}_i^2|\Psi_{MF}\rangle - \langle \Psi_{MF}|\hat{n}_i|\Psi_{MF}\rangle^2}{\langle \Psi_{MF}|\hat{n}_i|\Psi_{MF}\rangle}.$$

   (We have the notation $\hat{n}_i = a_i^\dagger a_i$.)

b) Explain briefly how you can distinguish between a Mott insulator and a superfluid phase based on the quantities above.

4. Consider the mean-field BCS Hamiltonian

$$H = \sum_{\mathbf{k}} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} & c_{-\mathbf{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\mathbf{k}} & \Delta \\ \Delta & -\xi_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} . \tag{2}$$

Diagonalize this by using the Bogoliubov transformation. Calculate the eigenenergies. You do not need to calculate the eigenvectors, we give them here:

$$u_{\mathbf{k}} = u_{\mathbf{k}\uparrow} = v_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)}, \tag{3}$$

$$v_{\mathbf{k}} = v_{\mathbf{k}\uparrow} = -u_{\mathbf{k}\downarrow} = \sqrt{\frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}}\right)}. \tag{4}$$

How are the new quasiparticle operators $\gamma_{\mathbf{k}\uparrow}$, $\gamma_{\mathbf{k}\downarrow}$ related to the original operators $c_{\mathbf{k}\uparrow}$, $c_{\mathbf{k}\downarrow}$? Calculate the momentum distribution $\langle n_{\mathbf{k}\uparrow}\rangle = \langle c^\dagger_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow}\rangle$ and the order parameter $\Delta = -\frac{V_0}{V}\sum_{\mathbf{k}} \langle c_{\mathbf{k}\uparrow} c_{-\mathbf{k}\downarrow}\rangle$ using the BCS ansatz

$$|BCS\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} c^\dagger_{\mathbf{k}\uparrow}) |0\rangle .$$

What is the functional dependence of $\langle n_{\mathbf{k}\uparrow}\rangle$ on $\mathbf{k}$ (e.g. sketch a figure) in the BCS and in the normal (non-interacting) state, and how does this relate to the concept of the Fermi surface? Note that $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ in the BCS ansatz are the same as in the Bogoliubov tranformation.