MS-E1993 Time-Frequency Analysis (Aalto University)
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Examination (Friday 26.5.2017, 13:00-16:00)

Points also for good effort! **Calculators and literature forbidden.**
Remark: The *Wigner distribution* $W[u] = W(u,u)$ is a special case of the *Wigner transform* $W(u,v)$ of nice-enough signals $u, v : \mathbb{R} \to \mathbb{C}$, defined by

$$W(u,v)(x,\eta) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}\, u(x+y/2)\, v(x-y/2)^*\, \mathrm{d}y. \qquad (1)$$

In your solutions, feel free to apply changes of variables, like

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(a,b)\, \mathrm{d}a\, \mathrm{d}b = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(x+y/2, x-y/2)\, \mathrm{d}x\, \mathrm{d}y.$$

1. How do we define the ambiguity transform? Show that the ambiguity transform is the symplectic Fourier transform of the Wigner transform.

2. a) Define the STFT (Short-Time Fourier Transform) of signal u with respect to window w. What is the corresponding spectrogram?

   b) Explain how the densities $|w|^2$ and $|\widehat{w}|^2$ show up in the spectrograms of δ_p, e_0, where δ_0 is the Dirac delta at time 0, and $e_0(x) \equiv 1$.

3. Let $C = \psi * W$ be a Cohen class time-frequency transform. The corresponding quantization $a \mapsto a_C$ is defined by $\langle u, a_C v\rangle := \langle C(u,v), a\rangle$.

   a) Suppose C has correct marginals in time (what does this mean?). Show that then $a_C v(x) = f(x)\, v(x)$ whenever $a(x,\eta) = f(x)$.

   b) Let $\phi = F\psi$ be the ambiguity kernel (where F is the symplectic Fourier transform) such that $|\phi| = 1$. Prove *Moyal's formula*

   $$\langle C[v], C[g]\rangle = |\langle v, g\rangle|^2.$$

4. The Born–Jordan transform $Q(u,v) = \psi * W(u,v)$ is defined by

   $$Q(u,v)(x,\eta) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta} \frac{1}{y} \int_{x-y/2}^{x+y/2} u(t+y/2)\, v(t-y/2)^*\, \mathrm{d}t\, \mathrm{d}y.$$

   Find a simple expression for $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, where

   $$a_Q v(x) = \int_{-\infty}^{\infty} K(x,y)\, v(y)\, \mathrm{d}y,$$

   where the symbol $a = \delta_{(0,0)}$ is the Dirac delta distribution at the origin of the time-frequency plane. (Do not leave integrals in your solution.)