Mathematics
Aalto University

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MS-E1651 - Numerical matrix computations

Exam 26.10.2017

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination.

You have two options

- Solve all problems. Grade is based only on the exam.
- Solve any three problems. Grade is based on exercise points and exam points (40/60).

1. Let $b \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n \times n}$. Consider the fixed-point iteration

$$x_k = \hat{b} + Cx_{k-1}. \tag{1}$$

Let x be the fixed point, this is, $x = \hat{b} + Cx$.

(a) Let $e_k = x - x_k$. Show that $e_k = C^{k-1}e_1$.

(b) For which $\alpha \in \mathbb{R}$ does iteration (1) converge, when

$$C = \frac{1}{\alpha}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{2}$$

The initial guess can be any $x_1 \in \mathbb{R}^3$.

2. Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 + \frac{1}{4} \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 5 + \frac{1}{4} \end{bmatrix}.$$

(a) Compute the Cholesky decomposition of A.

(b) Using the Cholesky decomposition, solve linear system $A\boldsymbol{x} = \boldsymbol{b}$.

(c) Derive a formula for computing the left-looking Cholesky decomposition of $n \times n$ symmetric and positive definite matrix.
(Hint: Write the matrices L and A in a block formulation and start with the condition $LL^T = A$.)

3. Let

$$A_1 := \begin{bmatrix} 1 & 0 & 2 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \\ 7 & 8 & 9 & 10 \end{bmatrix}. \tag{3}$$

and

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N = 5;
A2 = 2*eye(N) + diag(-ones(N-1,1),1)+ diag(-ones(N-1,1),-1)
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(a) Write A_1 and A_2 using the compressed column storage scheme.

(b) Explain how row and column access differ in compressed column storage scheme.

4. Consider solving the linear system $A\boldsymbol{x} = \boldsymbol{b}$ using the conjugate gradient method.

(a) Given iterate $\boldsymbol{x}^i$, residual $\boldsymbol{r}^{i+1}$ and search direction $\boldsymbol{p}^{i+1}$, derive a formula for computing iterate $\boldsymbol{x}^{i+1}$.

(b) Given search direction $\boldsymbol{p}_i$ and residual $\boldsymbol{r}^{i+1}$, derive a formula for computing search direction $\boldsymbol{p}_{i+1}$.

(c) Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In addition, let

$$\boldsymbol{x}^i = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \boldsymbol{r}^{i+1} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{p}^i = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Compute $\boldsymbol{p}^{i+1}$ and $\boldsymbol{x}^{i+1}$.