MS-E1461 Hilbert spaces (Aalto University)
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Examination (Monday 23.10.2017, 9:00–12:00)

Points also for good effort! **Calculators and literature forbidden.**

Remark. In the following problems, H is a complex Hilbert space.

1. Formulate and prove the Cauchy–Schwarz inequality in H.

2. a) How do we define orthogonal projection P_M onto a closed vector subspace $M \subset H$? (No proof needed!)

   b) Let M be the 1-dimensional vector subspace M spanned by $u \in H$, where $u \neq 0$. Find expressions for the orthogonal projections onto M and onto its orthogonal complement $M^\perp$. (Justify your answer!)

3. Let $H = L^2(\mathbb{R}^+)$, where $\mathbb{R}^+$ is the positive real line. Show that the Laplace transform $\mathscr{L}$ is bounded on H, where for nice $u : \mathbb{R} \to \mathbb{C}$

   $$\mathscr{L}u(x) := \int_0^\infty \mathrm{e}^{-xy}\, u(y)\, \mathrm{d}y.$$

   Hint: Calculate first, and justify only later in retrospect:

   $$\|\mathscr{L}\|^2 = \sup_{u \in H : \|u\| \leq 1} \langle \mathscr{L}^* \mathscr{L} u, u \rangle = \sup_{u \in H : \|u\| \leq 1} \langle \mathscr{L}^2 u, u \rangle = \ldots$$

   and apply the Hilbert inequality

   $$\int_0^\infty \int_0^\infty \frac{|u(x)\, v(y)|}{x+y}\, \mathrm{d}x\, \mathrm{d}y \leq \pi \left(\int_0^\infty |u(x)|^2\, \mathrm{d}x \int_0^\infty |v(y)|^2\, \mathrm{d}y \right)^{1/2}.$$

4. Think of a direct sum decomposition $H = \bigoplus_{\alpha \in J} H_\alpha$, and let $P_\alpha : H \to H$ be the orthogonal projection onto H_α. Suppose $|\varphi(\alpha)| < 1$ for all $\alpha \in J$, where $\varphi : J \to \mathbb{C}$ is injective. For $u \in H$, let

   $$Au := \sum_{\alpha \in J} \varphi(\alpha)\, P_\alpha u.$$

   a) Show that $A \in \mathscr{B}(H)$. When would A be self-adjoint?
   b) What is the spectrum $\sigma(A)$ of A here?
   Which are the eigenvalues and the corresponding eigenvectors?
   Can $\sigma(A)$ here contain something else than the eigenvalues?
   (Justify your answers! You may use the fact that $\sigma(A)$ is closed.)