MS-E1461 Hilbert spaces (Aalto University)
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Examination (Monday 21.10.2019, 9:00-12:00)

Points also for good effort! **Calculators and literature forbidden.**
If you need to use some results from the course, please clearly state them.

Remark. In the following, H is an infinite-dimensional **complex** Hilbert space, with inner product $(u,v) \mapsto \langle u,v\rangle$ and norm $u \mapsto \|u\| = \langle u,u\rangle^{1/2}$.

1. Formulate and prove the Cauchy–Schwarz inequality in H.

2. (a) Show that $u \mapsto \langle u,v\rangle$ is a bounded linear functional, of norm $\|v\|$.

   (b) What does the Fréchet–Riesz Representation Theorem say?

   (c) Let $\varphi : H \to \mathbb{C}$ be a bounded linear functional, and let $f(\alpha) := \varphi(e_\alpha)$, where $(e_\alpha)_{\alpha\in J}$ is an orthonormal basis of H. Show that $f : J \to \mathbb{C}$ belongs to $\ell^2(J)$, and that $\|f\| = \|\varphi\|$.

3. For $u \in \ell^2 := \ell^2(\mathbb{Z}^+)$ and $N \in \mathbb{Z}^+$, define $P_N u : \mathbb{Z}^+ \to \mathbb{C}$ by

$$P_N u(k) := \begin{cases} u(k) & \text{if } \ k \le N, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Let $P := P_N$. Show that $P \in \mathscr{B}(\ell^2)$ such that $P^2 = P = P^*$ (meaning that P is an orthogonal projection).

   (b) Show that $\|P_N u - u\| \to 0$ as $N \to \infty$, for all $u \in \ell^2$. Show that $\|P_N - I\| = 1$ for all $N \in \mathbb{Z}^+$.

4. Let $A : H \to H$ be a compact linear operator.

   (a) Explain why A^*A is a compact positive operator. What does the diagonalization of A^*A mean?

   (b) Show that A can be presented by

$$Av = \sum_{k=1}^{\infty} \sigma_k \langle v, v_k\rangle\, u_k$$

   for all $v \in H$, where $(u_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$ are orthonormal sequences, and $\sigma_k \ge \sigma_{k+1} \ge 0$ for all $k \in \mathbb{Z}^+$.

   (In other words, construct the Singular Value Decomposition (SVD).)