ELEC-E8101 Digital and Optimal Control

Intermediate Exam (27.10.2017)

- *Write your name and student number to each answer sheet.*
- *All four (4) problems must be answered.*
- *A datasheet is included in the back of the exam.*
- *Electronic calculators can be used. No calculators with capabilities for storing and displaying verbal information are allowed.*

(Turn over)

1. a) Find the z-transform of the sequence $x[k] = e^{-akh}$, $k = 0, 1, 2, 3, \ldots$, where h and a are constants, using the definition. [1p]

   b) Given that

$$Y(z) = \frac{(1 - e^{-ah})z}{(z-1)(z - e^{-ah})}, \quad a, h \text{ are constants,}$$

   find the value of $y[k]$ as $k \to \infty$ using the Final Value Theorem. [2p]

   c) Find $y[k]$ by doing the inverse z-transform of $Y(z)$ given above. Determine the value of $y[k]$ as $k \to \infty$ and check if your answer agrees with that of part b). [3p]

2. Consider the following difference equation:

$$y[k+2] - 1.3y[k+1] + 0.4y[k] = u[k+1] - 0.4u[k].$$

   a) Determine the pulse transfer function. [2p]

   b) Is the system stable? Justify your answer. [2p]

   c) Determine the step response. [2p]

3. The double integrator is a common process in mechanical models. Its differential equation form is

$$\frac{d^2y(t)}{dt^2} = u(t).$$

   a) Show that the state-space representation is given by [2p]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

   b) Sample the state-space model with sampling time h, assuming ZOH and determine the discrete state-space representation of the form: [2p]

$$\mathbf{x}(kh + h) = \Phi(h)\mathbf{x}(kh) + \Gamma(h)\mathbf{u}(kh)$$
$$\mathbf{y}(kh) = C\mathbf{x}(kh) + D\mathbf{u}(kh)$$

   Hint:

$$\Phi(h) = e^{Ah} = I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}h^nA^n$$

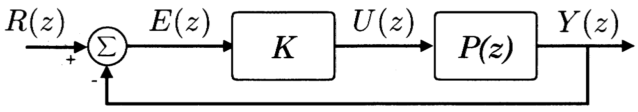
$$\Gamma(h) = \int_0^h e^{As} ds B$$

   c) Find the transfer function of the discrete-time representation. [2p]

   Hint: The transfer function is given by $G(z) = C(zI - \Phi)^{-1}\Gamma + D$.

(Turn over)

4. Consider the feedback system



where

$$P(z) = \frac{-1}{z^2 + z + 2}$$

and K is a constant.

a) Draw the pole/zero diagram (z-plane) for the *open-loop* system $P(z)$. Is the system stable? [2p]

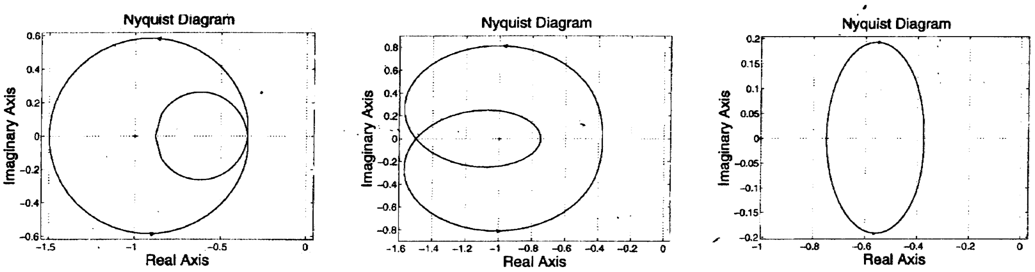
b) Show that the closed-loop transfer function from $R(z)$ to $Y(z)$ is given by [1p]

$$G(z) = \frac{-K}{z^2 + z + 2 - K}$$

c) For which values of K is the closed-loop stable? [3p]

d) Consider the closed-loop system and let the input $r[k]$ be a unit step. Find, as a function of gain K, the steady-state value of $y[k]$ (i.e., the $\lim_{k\to\infty} y[k]$) when this is finite, stating for which values of K the answer is valid. [3p]

e) Let $K = 1.5$. The figure below shows three Nyquist plots (A, B and C), but only one corresponds to $KP(z)$.



A **B** **C**

Choose the correct one, justifying your answer with respect to the Nyquist stability criterion. [3p]

Hint: The closed-loop system will be stable if and only if the number of counter-clockwise encirclements N of the point -1 by $KP(e^{j\omega})$ as ω increases from 0 to 2π is such that $N = Z - P$, where Z is the number of roots of the characteristic equation, $1 + KP(z) = 0$, outside the unit circle, and, P the number of roots of the open-loop system, $KP(z) = 0$, outside the unit circle.

TABLE OF LAPLACE TRANSFORMS

Waveform: $g(t)$ (defined for $t \geq 0$)	Laplace Transform: $G(s) = \mathcal{L}\{g(t)\} = \int_{0_-}^{\infty} g(t)e^{-st}dt$	
$\delta(t)$ impulse	1	
$u(t)$ unit step	$\frac{1}{s}$	
t^n	$\frac{n!}{s^{n+1}}$	
e^{-at}	$\frac{1}{s+a}$	
$\sin(\omega_0 t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	
$\cos(\omega_0 t)$	$\frac{s}{s^2+\omega_0^2}$	
$\sinh(\omega_0 t)$	$\frac{\omega_0}{s^2-\omega_0^2}$	
$\cosh(\omega_0 t)$	$\frac{s}{s^2-\omega_0^2}$	
$e^{-at}[A\cos(\omega_0 t) + B\sin(\omega_0 t)]$	$\frac{A(s+a)+B\omega_0}{(s+a)^2+\omega_0^2}$	
$e^{-at}g(t)$	$G(s+a)$	shift in s
$g(t-\tau)u(t-\tau)$ where $\tau \geq 0$	$e^{-s\tau}G(s)$	shift in t
$tg(t)$	$-\frac{d}{ds}G(s)$	
$\frac{dg}{dt}$ differentiation	$sG(s) - g(0)$	
$\frac{d^n g}{dt^n}$	$s^n G(s) - s^{n-1}g(0) - s^{n-2}\left(\frac{dg}{dt}\right)_0 - \cdots - \left(\frac{d^{n-1}g}{dt^{n-1}}\right)_0$	
$\int_0^t g(\tau)d\tau$ integration	$\frac{G(s)}{s}$	
$g_1(t) * g_2(t) = \int_0^t g_1(t-\tau)g_2(\tau)d\tau$ convolution	$G_1(s)G_2(s)$	

(Turn over)

TABLE OF Z-TRANSFORMS

Sequence: $g_k, \quad k = 0, 1, 2, \ldots$	Right-sided z-transform: $G(z) = \sum_{k=0}^{\infty} g_k z^{-k}$
1 (unit step)	$\frac{1}{1 - z^{-1}}$
kT	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$
$\frac{(k + m - 1)!}{k!(m - 1)!}$	$\frac{1}{(1 - z^{-1})^m}$
e^{-akT}	$\frac{1}{1 - e^{-aT}z^{-1}}$
$\sin(\omega_0 kT)$	$\frac{\sin(\omega_0 T)z^{-1}}{1 - 2\cos(\omega_0 T)z^{-1} + z^{-2}}$
$\cos(\omega_0 kT)$	$\frac{1 - \cos(\omega_0 T)z^{-1}}{1 - 2\cos(\omega_0 T)z^{-1} + z^{-2}}$
$\frac{r^{k-1}}{\sin \omega_0 T}[r \sin(\omega_0(k + 1)T) - a \sin(\omega_0 kT)]$	$\frac{1 - az^{-1}}{1 - 2r\cos(\omega_0 T)z^{-1} + r^2 z^{-2}}$
$r^k[A\cos(\omega_0 kT) + B\sin(\omega_0 kT)]$	$\frac{A + rz^{-1}(B\sin(\omega_0 T) - A\cos(\omega_0 T))}{1 - 2r\cos(\omega_0 T)z^{-1} + r^2 z^{-2}}$
$r^k g_k$	$G(r^{-1}z)$
g_{k+1}	$zG(z) - zg_0$
g_{k-1}	$z^{-1}G(z) + g_{-1}$
g_{k+m}	$z^m G(z) - z^m g_0 - \cdots - zg_{m-1}$
g_{k-m}	$z^{-m}G(z) + z^{-(m-1)}g_{-1} + \cdots + g_{-m}$
$g_0 = \lim_{z\to\infty} G(z)$	(initial value theorem)
$\lim_{k\to\infty} g_k = \lim_{z\to 1}(z - 1)G(z)$	(final value theorem when poles of $(z - 1)G(z)$ are inside unit circle)