ELEC-E8101 Digital and Optimal Control

Intermediate Exam (27.10.2017) - Solutions

1. a) Find the z-transform of the sequence $x[k] = e^{-akh},\ k = 0, 1, 2, 3, \ldots$, where h and a are constants, using the definition. [1p]

   b) Given that

$$Y(z) = \frac{(1 - e^{-ah})z}{(z-1)(z - e^{-ah})},\quad a, h \text{ are constants,}$$

   find the value of $y[k]$ as $k \to \infty$ using the Final Value Theorem. [2p]

   c) Find $y[k]$ by doing the inverse z-transform of $Y(z)$ given above. Determine the value of $y[k]$ as $k \to \infty$ and check if your answer agrees with that of part b). [3p]

Solution.

a) The z-transforms of the sequences based on the definition is

$$\begin{aligned}
X(z) &= \sum_{-\infty}^{\infty} x[k] z^{-k} \\
&= \sum_{k=0}^{\infty} e^{-akh} z^{-k} \\
&= \sum_{k=0}^{\infty} \left(e^{-ah} z^{-1}\right)^k \\
&= \lim_{k\to\infty} \frac{1 - \left(e^{-ah} z^{-1}\right)^k}{1 - e^{-ah} z^{-1}} \\
&= \frac{1}{1 - e^{-ah} z^{-1}}, \quad |e^{-ah} z^{-1}| < 1 \text{ or } |z| > |e^{-ah}| \\
&= \frac{z}{z - e^{-ah}}
\end{aligned}$$

b) *Final Value Theorem:* if $\lim_{k\to\infty} y[k]$ exists, then:

$$\begin{aligned}
\lim_{k\to\infty} y[k] &= \lim_{z\to 1} (1 - z^{-1}) Y(z) \\
&= \lim_{z\to 1} (z-1) \frac{(1 - e^{-ah})z}{(z-1)(z - e^{-ah})} \\
&= \lim_{z\to 1} \left(\frac{1 - e^{-ah}}{z - e^{-ah}} \right) \\
&= \frac{1 - e^{-ah}}{1 - e^{-ah}} = 1
\end{aligned}$$

c) If z-transform tables are studied, it is noted that in every transform z is in the numerator. Therefore, for convenience, we first divide the given equation by z,

$$\frac{Y(z)}{z} = \frac{\left(1 - e^{-ah}\right)}{(z-1)\left(z - e^{-ah}\right)} = \frac{A}{z-1} + \frac{B}{z - e^{-ah}}$$

Let's solve for A and B with Heaviside's method. (The partial fraction method could as well be used)

$$A = \lim_{z \to 1} (z-1) \frac{\left(1 - e^{-ah}\right)}{(z-1)\left(z - e^{-ah}\right)} = 1$$

$$B = \lim_{z \to e^{-ah}} \left(z - e^{-ah}\right) \frac{\left(1 - e^{-ah}\right)}{(z-1)\left(z - e^{-ah}\right)} = -1.$$

Then

$$\frac{Y(z)}{z} = \frac{1}{z-1} - \frac{1}{z - e^{-ah}} \Rightarrow Y(z) = \frac{z}{z-1} - \frac{z}{z - e^{-ah}}$$

and this can be inverse-transformed with the transformation tables, giving

$$y[k] = 1 - e^{-akh}.$$

Therefore,

$$\lim_{k \to \infty} y[k] = \lim_{k \to \infty} \left(1 - e^{-akh}\right) = 1.$$

The answer agrees with part b).

2. Consider the following difference equation:

$$y[k+2] - 1.3y[k+1] + 0.4y[k] = u[k+1] - 0.4u[k].$$

a) Determine the pulse transfer function. [2p]

b) Is the system stable? Justify your answer. [2p]

c) Determine the step response. [2p]

Solution.

a) Taking the z-transform (assuming zero initial conditions):

$$z^2Y(z) - 1.3zY(z) + 0.4Y(z) = zU(z) - 0.4U(z)$$
$$G(z) = \frac{Y(z)}{U(z)} = \frac{z-0.4}{z^2 - 1.3z + 0.4}$$

b) We want to find the poles of the transfer function, i.e.,

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z-0.4}{z^2-1.3z+0.4} = \frac{z-0.4}{(z-0.8)(z-0.5)}.$$

The poles are: $p_1 = 0.8$ and $p_2 = 0.5$. The poles are within the unit circle and therefore, the system is stable.

c) From the difference equation:

<u>Up to $k = -3$:</u> $\ldots = y[-3] = y[-2] = y[-1] = 0$.

<u>For $k = -2$:</u>

$$y[0] - 1.3\underbrace{y[-1]}_{=0} + 0.4\underbrace{y[-2]}_{=0} = \underbrace{u[-1]}_{=0} - 0.4\underbrace{u[-2]}_{=0} \Rightarrow y[0] = 0$$

<u>For $k = -1$:</u>

$$y[1] - 1.3\underbrace{y[0]}_{=0} + 0.4\underbrace{y[-1]}_{=0} = \underbrace{u[0]}_{=1} - 0.4\underbrace{u[-1]}_{=0} \Rightarrow y[1] = 1$$

Taking the z-transform (*without* assuming zero initial conditions):

$$\left(z^2Y(z) - z^2\underbrace{y[0]}_{=0} - z\underbrace{y[1]}_{=1}\right) - 1.3\left(zY(z) - z\underbrace{y[0]}_{=0}\right) + 0.4Y(z) = \left(zU(z) - z\underbrace{u[0]}_{=1}\right) - 0.4U(z)$$
$$z^2Y(z) - \not{z} - 1.3zY(z) + 0.4Y(z) = zU(z) - \not{z} - 0.4U(z)$$

Therefore, $Y(z)$ is given by

$$Y(z) = \frac{z-0.4}{z^2-1.3z+0.4}U(z) = \frac{z-0.4}{(z-0.8)(z-0.5)}U(z)$$
$$= \frac{z-0.4}{(z-0.8)(z-0.5)}\frac{z}{z-1} = \frac{z(z-0.4)}{(z-0.8)(z-0.5)(z-1)}$$

We do partial fractions:

$$\frac{Y(z)}{z} = \frac{z-0.4}{(z-0.8)(z-0.5)(z-1)} \equiv \frac{A}{z-0.8} + \frac{B}{z-0.5} + \frac{C}{z-1}$$

$$\left.\begin{aligned} A &= \frac{0.4}{0.3(-0.2)} = -\frac{20}{3} = -6.67 \\ B &= \frac{0.1}{(-0.3)(-0.5)} = \frac{2}{3} = 0.67 \\ C &= \frac{0.6}{(0.2)(0.5)} = 6 \end{aligned}\right\} \Rightarrow Y(z) = -6.67\frac{z}{z-0.8} + 0.67\frac{z}{z-0.5} + 6\frac{z}{z-1}$$

Therefore,

$$y[k] = -6.67(0.8)^k - 0.67(0.5)^k + 6u[k].$$

3. The double integrator is a common process in mechanical models. Its differential equation form is

$$\frac{d^2y(t)}{dt^2} = u(t).$$

a) Show that the state-space representation is given by [2p]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

b) Sample the state-space model with sampling time h, assuming ZOH and determine the discrete state-space representation of the form: [2p]

$$\mathbf{x}(kh+h) = \Phi(h)\mathbf{x}(kh) + \Gamma(h)\mathbf{u}(kh)$$
$$\mathbf{y}(kh) = C\mathbf{x}(kh) + D\mathbf{u}(kh)$$

Hint:

$$\Phi(h) = e^{Ah} = I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} h^n A^n$$
$$\Gamma(h) = \int_0^h e^{As} ds B$$

c) Find the transfer function of the discrete-time representation. [2p]

Hint: The transfer function is given by $G(z) = C(zI - \Phi)^{-1}\Gamma + D$.

Solution.

a) Set $x_1 = y$, $x_2 = dy/dt$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then,

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{d^2y}{dt^2} = u(t) \end{aligned}\right\} \Rightarrow \begin{cases} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

b) We need to find $\Phi(h)$ and $\Gamma(h)$:

$$\begin{aligned} \Phi(h) &= e^{Ah} = I + hA + \frac{1}{2}h^2A^2 + \frac{1}{6}h^3A^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} h^n A^n \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + h \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathbf{0} = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \\ \Gamma(h) &= \int_0^h e^{As} ds B = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} ds \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \int_0^h \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ds \\ &= \int_0^h \begin{bmatrix} s \\ 1 \end{bmatrix} ds = \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \end{aligned}$$

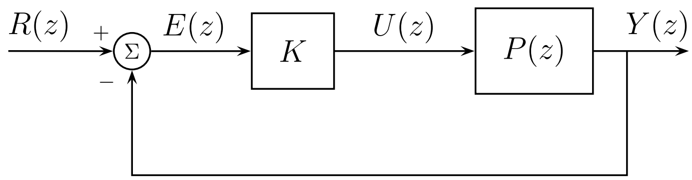
Therefore,

$$
\begin{aligned}
\mathbf{x}(kh+h) &= \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \mathbf{x}(kh) + \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \mathbf{u}(kh) \\
\mathbf{y}(kh) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(kh)
\end{aligned}
$$

c) The transfer function is given by

$$
\begin{aligned}
G(z) = C(zI-\Phi)^{-1}\Gamma + D &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z-1 & -h \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{(z-1)^2} \begin{bmatrix} z-1 & h \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} h^2/2 \\ h \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{z-1} & \frac{h}{(z-1)^2} \end{bmatrix} \begin{bmatrix} h^2/2 \\ h \end{bmatrix} = \frac{h^2/2}{z-1} + \frac{h^2}{(z-1)^2} = \frac{h^2(z+1)}{2(z-1)^2}
\end{aligned}
$$

4. Consider the feedback system



where

$$P(z) = \frac{-1}{z^2 + z + 2}$$

and K is a constant.

a) Draw the pole/zero diagram (z-plane) for the *open-loop* system $P(z)$. Is the system stable? [2p]

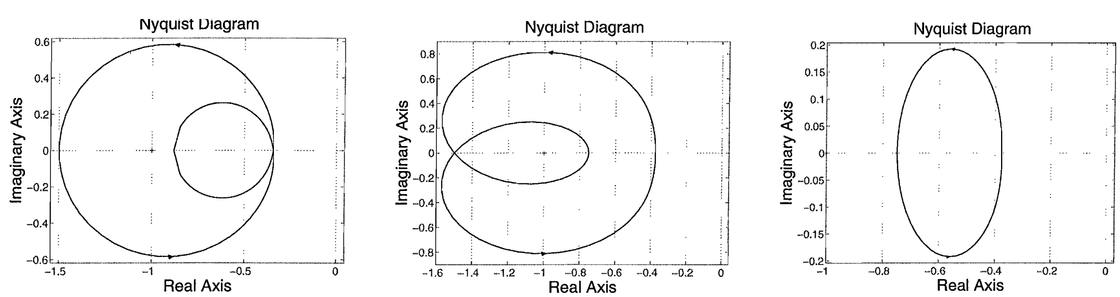
b) Show that the closed-loop transfer function from $R(z)$ to $Y(z)$ is given by [1p]

$$G(z) = \frac{-K}{z^2 + z + 2 - K}$$

c) For which values of K is the closed-loop stable? [3p]

d) Consider the closed-loop system and let the input $r[k]$ be a unit step. Find, as a function of gain K, the steady-state value of $y[k]$ (i.e., the $\lim_{k\to\infty} y[k]$) when this is finite, stating for which values of K the answer is valid. [3p]

e) Let $K = 1.5$. The figure below shows three Nyquist plots (A, B and C), but only one corresponds to $KP(z)$.



A **B** **C**

Choose the correct one, justifying your answer with respect to the Nyquist stability criterion. [3p]

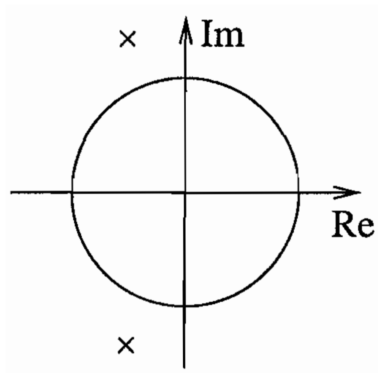
Hint: The closed-loop system will be stable if and only if the number of counter-clockwise encirclements N of the point -1 by $KP(e^{j\omega})$ as ω increases from 0 to 2π is such that $N = Z - P$, where Z is the number of roots of the characteristic equation, $1 + KP(z) = 0$, outside the unit circle, and, P the number of roots of the open-loop system, $KP(z) = 0$, outside the unit circle.

Solution.

a) The open-loop poles of the system are the roots of the equation $z^2 + z + 2 = 0$, i.e.,

$$p_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm j\sqrt{7}}{2}$$

The poles are outside the unit circle (see figure below), since $|p_{1,2}| > 1$, and therefore the system is unstable.



b) The closed-loop transfer function from $R(z)$ to $Y(z)$ is given by

$$G(z) = \frac{Y(z)}{R(z)} = \frac{KP(z)}{1 + KP(z)} = \frac{-K}{z^2 + z + 2 - K}.$$

c) **1st way:** The closed-loop poles are the roots of the equation $z^2 + z + 2 - K = 0$, which are given by

$$p_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4(1)(2 - K)}}{2(1)} = \frac{-1 \pm \sqrt{4K - 7}}{2}.$$

For closed-loop stability we need the poles to be inside the unit disk.

<u>For $4K - 7 < 0$:</u>

$$\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{4K - 7}}{2}\right)^2 < 1 \Rightarrow |4K - 7| < 3$$

1) Since we assume already that $4K - 7 < 0$, it holds that $4K - 7 < 3$. Hence, $K < 7/4$.
2) $-3 < 4K - 7 \Rightarrow K > 1$.

Therefore, for $4K - 7 < 0$, $1 < K < 7/4$.

<u>For $4K - 7 > 0$:</u>

$$-1 < \frac{-1 \pm \sqrt{4K - 7}}{2} < 1$$

which gives $7/4 < K < 2$.

So, **combining both cases**, $1 < K < 2$.

2nd way: Let's use the *Jury's stability test*:

$$\begin{array}{ccc|c} \boxed{1} & 1 & 2-K & \\ 2-K & 1 & 1 & b_2 = \dfrac{2-K}{1} = 2-K \\ \hline \boxed{1-(2-K)^2} & K-1 & & \\ K-1 & 1-(2-K)^2 & & b_1 = \dfrac{K-1}{1-(2-K)^2} \\ \hline 1-(2-K)^2 - \dfrac{(K-1)^2}{1-(2-K)^2} & & & \end{array}$$

The last term can be written as:

$$\begin{aligned} 1-(2-K)^2 - \frac{(K-1)^2}{1-(2-K)^2} &= (K-1)(3-K) - \frac{(K-1)^2}{(K-1)(3-K)} \quad \text{(difference of two squares)} \\ &= \frac{(K-1)^2(3-K)^2 - (K-1)^2}{(K-1)(3-K)} \\ &= \boxed{\frac{(K-1)^2\left[(3-K)^2-1\right]}{1-(2-K)^2}} \end{aligned}$$

Stability conditions require that the boxed expressions are all greater than 0. First, $1 > 0$ holds. For the second to hold we need:

$$\begin{aligned} 1-(2-K)^2 > 0 \Rightarrow & [1-(2-K)][1+(2-K)] > 0 \\ & (K-1)(3-K) > 0 \Rightarrow 1 < K < 3 \end{aligned}$$

For the third case, since the denominator is positive already (given that $1 < K < 3$ we want to make sure that $(3-K)^2 - 1 > 0$, which corresponds to: $K < 2$ or $K > 4$. Combining the two cases, we have that $\boxed{1 < K < 2}$.

3rd way: Using the triangle rule:

$$\begin{cases} -1 < 2-K < 1 \Rightarrow 1 < K < 3 \\ 0 < 2-K \Rightarrow K < 2 \\ -2 < 2-K \Rightarrow K < 4 \end{cases}$$

The solution is the intersection of the 3 sets given using the triangle rule, i.e., $\boxed{1 < K < 2}$.

d) When $K \notin (1,2)$, the system is unstable and therefore $y[k]$ will grow unbounded. When $k \in (1,2)$, the closed-loop system is stable and to find the steady-state value of $y[k]$, denoted here by y_{ss}, we use the Final Value Theorem to the closed-loop transfer function $G(z)$ we found in part b):

$$\begin{aligned} y_{ss} &= \lim_{k\to\infty} y[k] = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}(z-1)G(z)U(z) \\ &= \lim_{z\to 1}(z-1)\frac{-K}{z^2+z+2-K}\frac{z}{z-1} = \frac{-K}{4-K} \end{aligned}$$

e) **1st way:** For $K = 1.5$, the closed-loop system is stable. Since the open-loop system has 2 unstable poles, the Nyquist diagram must have 2 counterclockwise encirclements of the point $-1 + j0$. Thus, plot B is the correct.

2nd way: Nyquist plot A shows that, for $z = 1$ or $z = -1$, $KP(z) = -1.5$. However, $KP(1) = -3/8$ and $KP(-1) = -3/4$, thus plot A cannot be the one. Nyquist plot C shows that the magnitude of $KP(z)$ is approximately always less than 0.75. However, $|KP(e^{j1.93})| = 1.6$. Also, there exists only one encirclement, and the system could never be stable. Therefore, plot C cannot be the one either. Plot B satisfies all of the above and it is the correct one.