MS-C2111 Stochastic processes	**Exam**
Department of Mathematics and Systems Analysis	13.12.2017, 16:30–19:30
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The exam consists of 4 problems, each worth 6 points. You are allowed to use a **calculator** approved by the Finnish Matriculation Examination Board and a handwritten **memory aid sheet** of size A4 with text only on one side and with your name and student number in the upper right corner. You don't need to return your memory aid sheet.

Bonus points (0–6) earned in the exercises of the 2017 course are added to the exam score. In particular, for a student who has earned full bonus points in the exercises, it suffices to solve only three problems of his/her choice in the exam to achieve the maximum course score.

1. Consider a discrete time Markov chain on the state space $S = \{1, 2, 3, 4\}$ and with transition probability matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{4}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}.$$

   (a) Draw the transition diagram of the Markov chain. **(1 p)**

   (b) Does the Markov chain have a unique stationary distribution? **(1 p)**

   (c) If the Markov chain is currently in state 2, then what is the probability that it will be in state 2 also after four time steps? **(2 p)**

   (d) Calculate the expected travel time to (=hitting time of) the state 4 for the chain started from state 1. **(2 p)**

2. Assume that during a meteor shower, amateur astronomer Stella observes shooting stars according to a Poisson process with intensity $\lambda = 0.45$ (in units $\frac{1}{\text{min}}$), with the counter process denoted by $N = (N(t))_{t \geq 0}$ (the unit of time t is minutes). Calculate the following quantities and justify your answers:

   (a) Probability $\mathbb{P}[N(15) = 5]$ that during a quarter of an hour, Stella observes exactly five shooting stars. **(1 p)**

   (b) Conditional probability $\mathbb{P}[N(20) = 10 \,|\, N(10) = 5]$ that Stella observes ten shooting stars in total during the first twenty minutes, given that during the first ten minutes she observed five of them. **(2 p)**

   (c) Conditional probability $\mathbb{P}[N(5) = 0 \,|\, N(30) = 12]$ that Stella did not observe any shooting stars during the first five minutes, given that during her entire session of half an hour she observed a dozen of them. **(2 p)**

   (d) The most likely number of shooting stars observed by Stella during the first 15 minutes. **(1 p)**

3. By sending identical letters to three people, Tarmo initiates a chain letter of good old times, in which the receipients are asked to send a copy of the letter further to three new receipients. Assume for simplicity that each receipient of the chain letter independently of others follows the letters request of further sending three copies with probability 0.40, and otherwise does not send any copies.

   (a) Write down the probability generating function ϕ_Y of the number Y of letters sent by one receipient. **(2 p)**

   (b) In expected value, how many people receive a letter from a person who received theirs from the initial sender Tarmo? **(2 p)**

   (c) What is the probability (according to the above simplified model) that the sending of the chain letter will continue forever? **(2 p)**

   Hint: Given a third degree polynomial equation, once one root can be identified, solving the equation may then be reduced to a second degree equation.

4. An airport shuttle makes tours to transport passengers from a train station to two possible terminals, A and B. After carrying all of its passengers to their destination terminals, the shuttle returns to the train station to pick up the next passengers. The tour of the shuttle takes:

   - 12 min, if the shuttle is carrying passengers headed to both terminals
   - 10 min, if the shuttle is carrying passengers headed to terminal B but none to terminal A
   - 8 min, if the shuttle is not carrying any passengers headed to terminal B (including if there are no passengers at all).

   New passengers headed to terminal A keep arriving at the train station with independent waiting times which follow $\mathrm{Exp}(\frac{1}{2\,\mathrm{min}})$ distribution, and new passengers headed to terminal B arrive with independent waiting times which follow $\mathrm{Exp}(\frac{1}{6\,\mathrm{min}})$ distribution, independently also of each other. Always after returning from its previous tour, the shuttle picks up all new passengers that have arrived meanwhile, and starts a new tour to carry them to their destination terminals.

   (a) Construct a discrete time three state Markov chain describing the tours of the shuttle, such that the states indicate whether the tour has passengers headed to both terminals, has passengers headed to terminal B but not to terminal A, or only has passengers headed to terminal A (including the case of no passengers at all). Justify briefly why the process is a Markov chain. Give the transition probability matrix of the Markov chain. **(3 p)**

   (b) In the long run, what is the average duration of the tours of the shuttle? **(3 p)**