PHYS-E0415 Statistical Mechanics

Final exam, December 14, 2017, 13.00-16.00

You should answer in English unless you have special permission to use another language. You are free to use the lecture notes, the articles, the home work exercises, electronic devices, etc. Please write your name, student number, study program, course code, and the date in all of your papers. There are 3 problems in this exam set which consists of 2 pages.

Problem 1

Answer the following questions in your own words. No calculations are needed. Less than one page should suffice to answer all three questions:

a) Discuss the heat capacity of metals. What is the role of electrons and phonons?

b) For a system in the canonical ensemble, the Helmholtz free energy reads $F = U - TS$ in terms of the energy U, the temperature T, and the entropy S. At low temperatures, the system will be in its ground state with the lowest energy U. Describe how a phase transition may occur as the temperature is increased.

c) Describe an experimental method to detect the tunneling of single electrons on and off a metallic island.

Problem 2

Consider a gas of indistinguishable, non-interacting quantum particles (either bosons or fermions) in a two-dimensional potential of the form $V(x, y) = m\omega_0^2(x^2+y^2)/2$. The single-particle energy levels are given by $\varepsilon(n_x, n_y) = \hbar\omega_0(n_x + n_y + 1)$, where $n_x, n_y = 0, 1, 2, \ldots$ are integers.

a) At large energies, show that the density of states is well-approximated by the expression

$$g(\varepsilon) \simeq \frac{\varepsilon}{(\hbar\omega_0)^2}.$$

Hint: It may be useful to consider the contours of constant energy in the (n_x, n_y)-plane.

b) The gas is in contact with a particle reservoir at temperature T and chemical potential μ. Show that the average number of particles (either bosons or fermions) in the gas reads

$$\langle N \rangle = \pm \left(\frac{k_B T}{\hbar\omega_0}\right)^2 \mathrm{Li}_\nu(\pm z),$$

where $z = e^{\mu/(k_B T)}$ is the fugacity and $\mathrm{Li}_\nu(z) = \sum_{n=1}^{\infty} z^n/n^\nu$ is the polylogarithm of order ν. Which sign ($\pm$) corresponds to bosons and fermions, and what is the order ν?

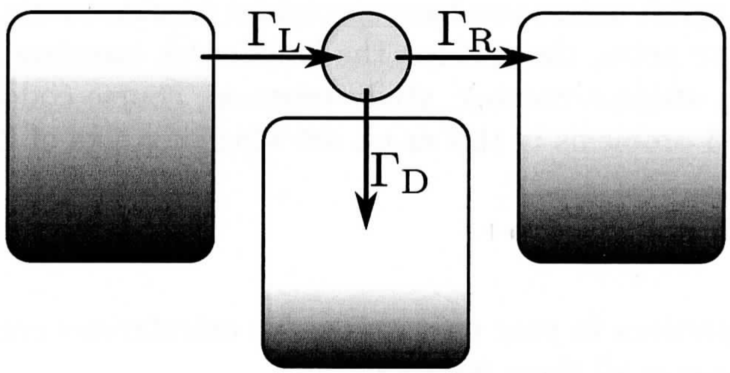
c) For bosons, show that the gas undergoes Bose-Einstein condensation at the temperature

$$T_c = \frac{\hbar\omega_0}{\pi k_B}\sqrt{6\langle N\rangle}.$$

Hint: It may be useful to know that $\mathrm{Li}_2(1) = \pi^2/6$, $\mathrm{Li}_2(-1) = -\pi^2/12$, or $\mathrm{Li}_4(1) = \pi^4/90$.

d) For $T < T_c$, find the fraction of particles in the ground state $n_0/\langle N\rangle$ depending on T/T_c.

Problem 3



The figure above shows three electrodes connected to a metallic island. The island can be either empty or occupied by one electron. Tunneling of electrons between the electrodes and the island occurs with the rates Γ_L, Γ_R, and Γ_D, in the directions indicated by the arrows.

a) Formulate rate equations for the probabilities $P_0(t)$ and $P_1(t)$ for the island to be either empty or occupied at time t.

We now consider the probabilities $P_N(n,t)$ that the island is in the charge state $N = 0, 1$, and $n \geq 0$ electrons have been collected in the right electrode during the time span $[0,t]$.

b) Formulate rate equations for the probabilities $P_0(n,t)$ and $P_1(n,t)$.

c) Define $P_N(\chi,t) = \sum_{n=0}^{\infty} P_N(n,t)e^{in\chi}$ and show they fulfill a matrix equation of the form

$$\frac{d}{dt}\begin{pmatrix} P_0(\chi,t) \\ P_1(\chi,t) \end{pmatrix} = \left[\begin{pmatrix} -\Gamma_L & \Gamma_R + \Gamma_D \\ \Gamma_L & -\Gamma_R - \Gamma_D \end{pmatrix} + (e^{i\chi} - 1)\mathbf{J}\right]\begin{pmatrix} P_0(\chi,t) \\ P_1(\chi,t) \end{pmatrix}.$$

Express the matrix $\mathbf{J}$ in terms of the tunneling rates.

d) Determine the cumulant generating function of the (particle) current $I \equiv n/t$ for $t \to \infty$.

e) Evaluate the first cumulant of I, $\langle\!\langle I \rangle\!\rangle$.

f) How would you find the second cumulant of I, $\langle\!\langle I^2 \rangle\!\rangle$? (no calculations needed)

g) In the course, we have learned that the distribution of waiting times between tunneling events into the right electrode can be expressed as

$$\mathcal{W}(\tau) = \frac{\langle\!\langle \tilde{0}|\mathbf{J}e^{\mathbf{M}_0\tau}\mathbf{J}|0\rangle\!\rangle}{\langle\!\langle \tilde{0}|\mathbf{J}|0\rangle\!\rangle}$$

Express $|0\rangle\!\rangle$ and $\mathbf{M}_0$ in terms of the tunneling rates Γ_L, Γ_R, and Γ_D.

End of exam set